MATHEMATICS MAGAZINE

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MATHEMATICS MAGAZINE

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ANNOUNCEMENT

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GEOMETRIC EXTREMUM PROBLEMS

G. D. CHAKERIAN, University of California, Davis, and
L. H. LANGE, San Jose State College

1. Introduction. A standard exercise for calculus classes reads: *Given a triangle of altitude a and base b, find the dimensions of the rectangle of maximum area which can be inscribed in this triangle with one side along the base.*

It at least broadens a student's perspective if he occasionally sees an alternate solution of such a problem avoiding the calculus. The above problem can be settled using an elementary inequality as follows:

Two essentially different possibilities face us, as shown in Figure I. In case (i), the vertex C is "above" some point of the base; in case (ii), the vertex C is not so situated. We solve case (i), and then the solution of case (ii) is easily made to depend on that solution.

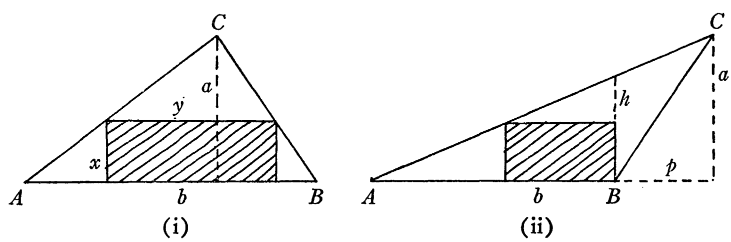

FIG. I.

Using the notation in Figure I, we seek the maximum of the product xy, where, from similar triangles, we have $y=(b/a)(a-x)$. Thus, $xy=(b/a)(x)(a-x)$, and we need to find the value of x, $0 \leqq x \leqq a$, which maximizes this quantity. Rather than use the derivative, we can do this directly by observing that $x(a-x)=(a/2)^2-\{x-(a/2)\}^2$, and this is a maximum if, and only if, $(x-a/2)^2=0$; i.e., iff $x=a/2$. Hence, $xy=(b/a)(x)(a-x) \leqq \frac{1}{2}(ab/2)=\frac{1}{2}$ area $(\triangle ABC)$, with equality holding iff $x=a/2$. Thus, the maximum rectangle has height $x=a/2$ and area exactly half that of the given triangle. In case (ii), the maximum rectangle will have height $h/2$ and area less than half that of the given triangle. (In this case, the maximum area is $\frac{1}{2}\{b/(b+p)\}$ area $(\triangle ABC)$.)

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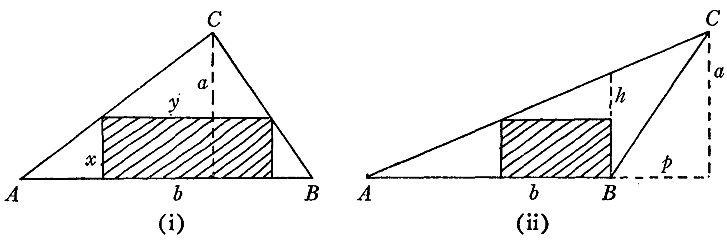

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Note that, in case (ii), if we choose AC as the base on which to place our rectangle, we would obtain a maximum rectangle with area half that of the given triangle. Hence, we observe that in any case, it is possible to inscribe *some* rectangle of area half that of the given triangle.

In calculus texts, it is common to treat, implicitly or explicitly, only those cases where the optimal figure is assumed to be in some special position. For example, in the problem above we considered only rectangles with a side lying on a base of the given triangle. But it is natural to inquire: of *all* rectangles contained in a given triangle, which yield the maximum area? Is this maximum area ever larger than half the area of the given triangle?

We shall answer this question in Section 2 (Theorem 3, below) and in later sections look at other extremum problems where the optimal figure is usually assumed to be in some restrictive special position. However, our main objective in this article is to provide an easily accessible account of a general class of geometric extremum problems of the above type, with examples which might be useful in the classroom (not necessarily only in calculus courses). An underlying theme, below, is the use of affine transformations to simplify problems of this type.

2. Polygons of minimum area circumscribed about a convex set. We shall deal with plane convex sets; that is, those plane sets having the property that the segment joining any two points of the set is contained in the set. By a *convex region* we shall mean a plane convex set which is also compact (i.e., closed and bounded), and has nonempty interior.

It is known that if K is a convex region and $n \geqq 3$ a given integer, then there exists at least one convex n-gon of minimum area containing K. Obviously, such an n-gon must be circumscribed about K—that is, its sides must intersect the boundary of K. The following theorem features the property which interests us:

THEOREM 1. *Let K be a convex region and $n \geqq 3$ a given integer. Let P be a convex n-gon of minimum area containing K. Then the midpoints of the sides of P lie on the boundary of K.*

Remark. Although this is well known (see [3] p. 6), we know of no easily accessible published proof, so we feel justified in presenting an elementary proof in Section 4. The proof will illustrate the usefulness of the technique of affine transformation developed in Section 3.

As a simple application of Theorem 1, consider the case where K is a parallelogram, and let T be a triangle of minimum area containing K. According to Theorem 1, the midpoints of the sides of T meet K. The reader will readily convince himself that this is possible only if one or two sides of K lie on sides of T, and the relative positions are as depicted in Figure II(a) or II(b).

Notice, in any case, that area $(T) = 2$ area (K). Keeping in mind that T was a minimal triangle, we thus have the following result:

THEOREM 2. *Let K be a given parallelogram, and let T be any triangle containing K. Then, area $(T) \geqq 2$ area (K), and equality holds if, and only if, T is in a special position, as depicted in Figure II.*

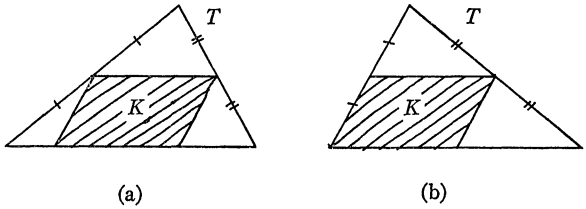

FIG. II.

We are now in a position to settle the question raised in Section 1. We have,

THEOREM 3. *Let T be a given triangle, and let R_0 be a rectangle of maximum area contained in T. Then, area $(R_0) = \frac{1}{2}$ area (T), and R_0 is in a special position, with one side lying on a side of T and the midpoints of the other two sides of T are vertices of R_0.*

Proof. Suppose R is a rectangle contained in T but such that R is not in the special position described above. Then, Theorem 2 implies that T is *not* a triangle of minimum area containing R; hence, if T^* *is* such a minimum triangle, we have

$$\text{area } (R) = \tfrac{1}{2} \text{ area } (T^*) < \tfrac{1}{2} \text{ area } (T).$$

If, on the other hand, R_0 is a rectangle in the special position described above, then area $(R_0) = \frac{1}{2}$ area (T). It follows that R_0 is a rectangle of maximum area contained in T. (Note that there are three such rectangles if T is an acute triangle, two if T is a right triangle, and only one if T has an obtuse angle.) This completes the proof.

Remarks. The problem which leads to Theorem 3 was raised in a paper by one of the authors [7]. In the meantime, M. T. Bird [1] has given a simple direct proof of Theorem 3.

Theorem 2 is contained in a result of Fulton and Stein [4, Theorem 1].

Parallelograms are rather extreme in their behavior with regard to circumscribed triangles—one cannot keep them inside a triangle of less than twice their area. One is naturally led to ask two questions:

(a) Are parallelograms the only convex regions which behave in this (deplorable) fashion?

(b) Is every convex region K contained in some triangle whose area is less than or equal to twice the area of K?

We will return to these questions after developing some tools, namely affine transformations, to help us simplify these and related questions.

In relation to Theorem 3, C. Radziszewski [9] proved that every convex region K contains a rectangle of half its area.

3. Affine transformations. In this section we call attention to some useful properties of nonsingular affine transformations of the plane—that is, those transformations of the (x, y)-plane onto itself which send each point (x, y) to a point (x', y') such that

$$x' = ax + by + e, \qquad y' = cx + dy + f,$$

where $a, b, \cdots, f$ are some given real numbers which satisfy the condition $ad - bc \neq 0$. Since we shall be interested only in nonsingular transformations, we shall consistently use the term "affine transformation" to mean *nonsingular* affine transformation.

Before listing those properties of affine transformations which we shall later find useful, we first look at an *instructive example* which involves a function of this type and the consideration of a certain kind of geometric extremum. (See [8] for an associated discussion.)

Letting a and b be given real numbers satisfying $0 < a \leqq b$, and letting $\mathcal{P}$ be the set of all pairs (x, y) of real numbers—our *plane*—we consider the function μ which maps $\mathcal{P}$ into $\mathcal{P}$ by sending the point (x, y) into the point $(x', y') = (ax, by)$. If we now consider the set D of all points (x, y) such that $x^2 + y^2 \leqq 1$, we see that the associated points (x', y') necessarily satisfy $(x'/a)^2 + (y'/b)^2 \leqq 1$.

Our function μ is a very special affine transformation; it is, in fact, a simple example of a *linear transformation*. The equations $x' = ax$ and $y' = by$ tell us that, given any (x', y') in $\mathcal{P}$, there exists a unique (x, y) in $\mathcal{P}$ which is such that μ sends (x, y) into (x', y'); that is, μ is a *one-to-one, onto* function. In particular, we see that each point of the (closed) elliptical disc shown in Figure III is the image of exactly one point in the (closed) circular disc D. The function μ "picks up the circular sheet D, and, without any folding (since μ is one-to-one) covers completely (since μ is onto) and exactly the elliptical disc $\mu(D)$."

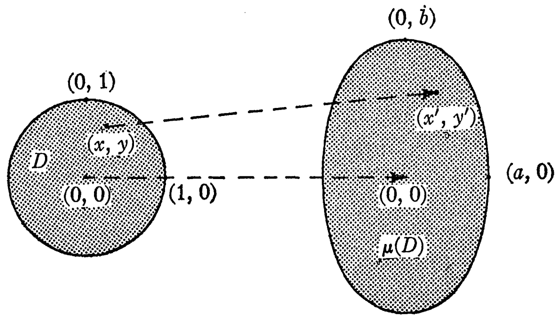

FIG. III.

Now suppose p, q, and r are given real numbers, and consider the set

$$\{(x, y) : px + qy + r = 0\},$$

which is a straight line. Then the function μ sends this set into the set

$$\left\{(x', y') : \frac{p}{a} x' + \frac{q}{b} y' + r = 0\right\}.$$

Thus, under μ, the image of any given straight line is again a straight line. We easily see that the image of the vertical line segment joining (x_1, y_1) and (x_1, y_2) is the vertical line segment which joins the image points (ax_1, by_1) and

(ax_1, by_2). See Figure IV. We notice also that the *length* of this image segment is simply $(b)\cdot|y_2-y_1|$, where $|y_2-y_1|$ is, of course, the length of the original segment. Similarly, the length of an image horizontal segment is simply (a) times the length of the original horizontal segment.

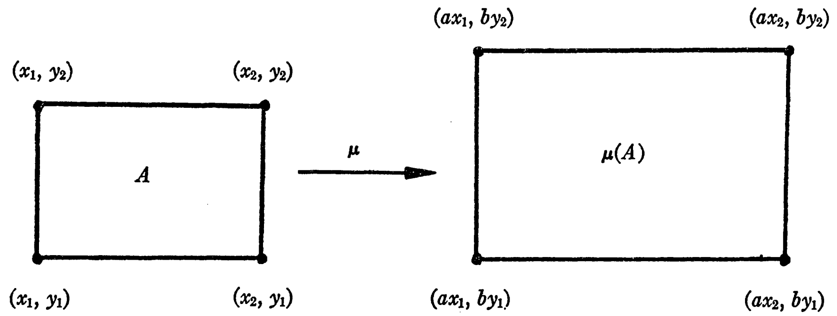

FIG. IV.

Now, if we again look at the image of the circular disc D under the function μ, we see that if the *rectangle* A in Figure IV is *inside* the disc D, then its image, $\mu(A)$ is a *rectangle* and it is *inside* the elliptical disc $\mu(D)$. Furthermore, if the area of A is α, then the area of the rectangle $\mu(A)$ is (ab) times α.

It follows that, if $\Sigma\alpha_i$ is the combined area of any finite collection of such rectangles inside D, then $(ab)\Sigma\alpha_i$ is the area of the associated collection of rectangles inside the elliptical disc $\mu(D)$.

Consequently, if we "pack" any finite number of such rectangles (of various sizes) into D, requiring, of course, that these rectangles don't "overlap"—other than along edges—we know that the sum of their areas, $\Sigma\alpha_i$, satisfies $\Sigma\alpha_i<4$, since 4 is the area of a square of side 2 enclosing D. Consequently, the sum of the areas of the associated rectangles in $\mu(D)$ satisfies $(ab)\Sigma\alpha_i<(ab)(4)$.

Now, the number 4 is not the *least* number which will serve as an upper bound for the collection of all possible numbers $\Sigma\alpha_i$ here. This distinction is, of course, held by the number π; and because π is the *least* upper bound of the collection of all such possible sums $\Sigma\alpha_i$, it is called the *area* of the disc D.

It follows that the number $(ab)\pi$ *must* be the area of the elliptical disc $(x/a)^2+(y/b)^2\leqq 1$, for it is the least number which will serve as an upper bound to the collection of all numbers $(ab)\Sigma\alpha_i$.

Exercise 1. Show that the volume of the ellipsoid

$$(x/a)^2 + (y/b)^2 + (z/c)^2 \leqq 1 \text{ is } \frac{4}{3}\pi abc.$$

We now list—without proof—those *properties of affine transformations* of the plane which we shall be using.

(P-1) If l and m are parallel lines, then their images are parallel lines. ("Parallel lines are sent to parallel lines.")

(P-2) If u^* and v^* are the images of line segments u and v respectively, lying on parallel lines, then

$$\{\text{length } (u^*)/\text{length } (v^*)\} = \{\text{length } (u)/\text{length } (v)\}.$$

("The ratio of lengths of parallel segments is preserved.")

In particular, we have

(P-3) If u^* is the image of the line segment u, then the midpoint of u^* is the image of the midpoint of u. ("Midpoints are sent into midpoints.")

(P-4) The image of any convex region is again a convex region.

(P-5) If U^* and V^* are the respective images of the regions U and V, then $\{\text{area } (U^*)/\text{area } (V^*)\} = \{\text{area } (U)/\text{area } (V)\}$. ("The ratio of areas is preserved.")

(P-6) Any given triangle can be mapped onto any other given triangle by some appropriate affine transformation. (Note that P-3 then implies that their centroids will correspond.)

(P-7) Any parallelogram can be mapped onto any other parallelogram by some affine transformation.

(P-8) The image of any ellipse is again an ellipse, and any ellipse can be mapped onto any other ellipse by some affine transformation. (Note that the center of an ellipse is sent to the center of the image. Indeed, P-3 implies that centrally symmetric figures are always sent into centrally symmetric figures, with centers corresponding.)

(P-9) Any (nonsingular) affine transformation is continuous and invertible, and the inverse is again a (nonsingular) affine transformation.

Here is a simple consequence of these properties:

LEMMA. *There is contained in any triangle T one and only one ellipse E_0 tangent to the sides of T at their respective midpoints.*

Proof. At least one such E_0 exists. For we may affinely transform T onto an equilateral triangle T^* and consider the inscribed circle (call it E_0^*) of T^*. The image of E_0^* under the inverse transformation is the required ellipse E_0.

Suppose F_0 were another ellipse inside T tangent to the sides of T at their midpoints. Then, the image F_0^* of F_0 under the above transformation would be an ellipse tangent to the sides of the equilateral triangle T^* at their midpoints. But the center of F_0^* coincides with the centroid of T^*. (In order to see this, map F_0^* to a circle and note that T^* must map to an *equilateral* triangle circumscribed about that circle—then use properties P-6 and P-8.) Hence, central reflection through the centroid of T^* sends F_0^* onto itself. It follows that F_0^* contains not only the midpoints of the sides of T^*, but also the midpoints of the sides of the reflection of T^* through its centroid. Since these midpoints are the vertices of a regular hexagon inscribed in E_0^*, and since an ellipse is determined by five points, it follows that $F_0^* = E_0^*$; hence, $F_0 = E_0$. This completes the proof.

The last lemma enables us to establish an extremum property analogous to Theorem 3.

THEOREM 4. *Any given triangle T contains a unique ellipse E_0 of maximum area. This ellipse is tangent to the sides of T at their midpoints, and area (E_0) $= (\pi/3\sqrt{3})$ area (T).*

Proof. Suppose E is an ellipse contained in T but such that the midpoints of the sides of T do not all meet E, and let E_0 be the unique ellipse inside T which is tangent to the sides of T at their midpoints. Further, let S be a triangle of minimum area containing E. Then, by Theorem 1, the midpoints of the sides of S lie on E, and area $(S) <$ area (T).

Now affinely map S onto T. Then E is mapped to an ellipse E^* in T tangent to the sides of T at their midpoints; hence, by the last lemma, $E^* = E_0$. By property P-5, we have then

$$\frac{\text{area }(E)}{\text{area }(S)} = \frac{\text{area }(E^*)}{\text{area }(T)} < \frac{\text{area }(E_0)}{\text{area }(S)},$$

hence, area $(E) <$ area (E_0). Thus, E_0 is the unique ellipse of maximum area contained in T. The ratio

$$\{\text{area }(E_0)/\text{area }(T)\} = \pi\sqrt{3}/9$$

is obtained by mapping T onto an equilateral triangle. This completes the proof.

REMARK. Theorem 4 illustrates a special case of the following general result: *Every convex region contains a unique ellipse of maximum area, and also is contained in a unique ellipse of minimum area.* For a discussion of this result and the generalization to higher dimensional spaces, the reader may consult [2].

Exercise 2. Use the methods of this section to prove that any given triangle T is contained in a unique ellipse E_1 of minimum area, and area $(E_1) = 4\pi/3\sqrt{3}$ area (T). [Hint: affinely map T onto an equilateral triangle T^* and let E_1^* be the circumcircle of T^*. The required E_1 is the image of E_1^* under the inverse transformation.]

Exercise 3. Using Exercise 2 and Theorem 4, prove that the circumradius of any triangle is at least twice the inradius. (For an elegant proof of this property, by I. Ádám, see [3] p. 28.)

4. Proof of Theorem 1. For the proof we need the following lemma, which tells us how to cut off a triangle of minimum area with a line passing through a given point inside an angle.

LEMMA. *Let XOY be a given angle* (Figure V). *Then,* (a) *for each point M interior to the angle there exists one and only one line segment $\overline{AB}$ containing M, with endpoints A on $\overrightarrow{OX}$ and B on $\overrightarrow{OY}$, bisected by M. Moreover,* (b) *of all triangles cut from the angle by lines passing through M, $\triangle AOB$ is the unique one of minimum area.* (c) *Let $\triangle(Q)$ denote the minimum triangle associated with a point Q interior to our angle XOY. If $Q \to M$ along $\overline{AB}$, then $\triangle(Q) \to \triangle(M)$. By this we mean the vertices of $\triangle(Q)$ tend to the corresponding vertices of $\triangle(M)$.*

Proof. A simple continuity argument shows that there is always *some* segment $\overline{AB}$ bisected by M. Now transform the configuration with an affine transformation sending A to $A^* = (1, 0)$, O to $O^* = (0, 0)$, and B to $B^* = (0, 1)$. Then automatically M is sent to $M^* = (\frac{1}{2}, \frac{1}{2})$. We can now establish all the properties in this situation. For this purpose we revert to the notation in our original figure, omitting the asterisks.

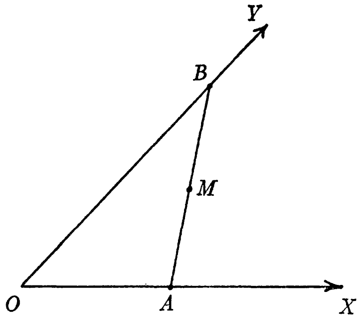

FIG. V.

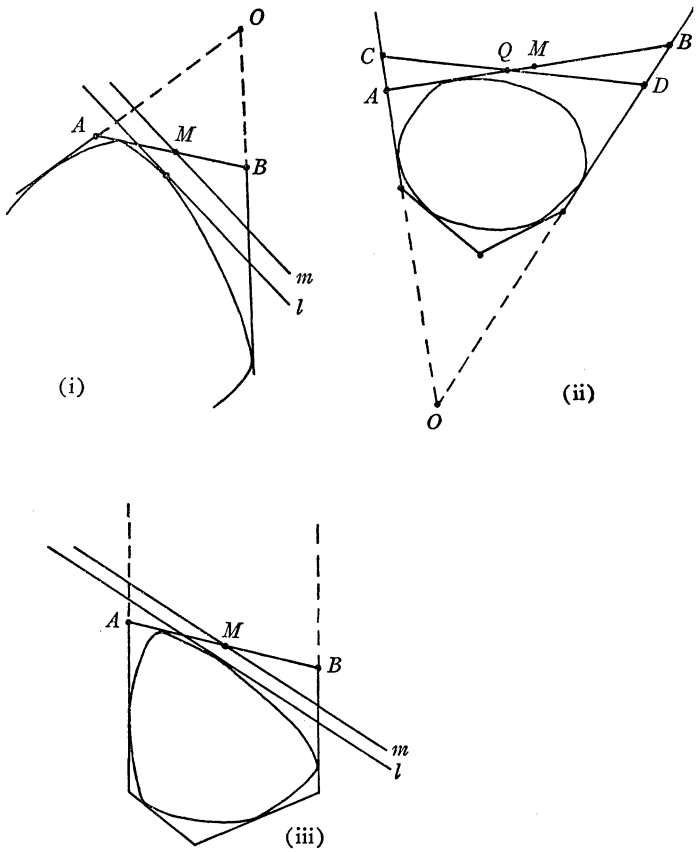

FIG. VI.

Indeed, let $Q=(\xi, 1-\xi)$, $0<\xi<1$, be any point in the interior of AB, and let $\triangle COD$ be any triangle cut off by a line passing through Q. Then, letting $C=(c, 0)$, we have

$$\text{area } (\triangle COD) = \frac{c^2(1-\xi)}{2(c-\xi)} \geqq 2\xi(1-\xi).$$

(The inequality is a consequence of $(2\xi-c)^2 \geqq 0$.) Equality holds if and only if $c=2\xi$. Hence, there is a unique minimizing triangle $\triangle(Q)$ with a side passing through Q and bisected by Q, and,

$$\text{area } (\triangle(Q)) = 2\xi(1-\xi).$$

If $A'OB'=\triangle(Q)$, then note that $A'=(2\xi, 0)$ and $B'=(0, 2-2\xi)$. Thus, as $Q\to M$ along $\overline{AB}$, we have $A'\to A=(1, 0)$ and $B'\to B(0, 1)$; that is $\triangle(Q)\to\triangle(M)$. The properties (a), (b), and (c) asserted in the theorem are now easily verified using the properties P-3, P-5, and P-9 of affine transformations in Section 3, applied to the inverse of the transformation we used above.

We can now prove Theorem 1. Suppose P is an n-gon of minimum area circumscribed about the convex region K, and suppose the midpoint M of some side $\overline{AB}$ does *not* meet K. We consider the three possible cases depicted in Figure VI.

In case (i) the sides of P adjacent to $\overline{AB}$, when extended, meet in a point O such that $\triangle AOB$ does not contain K. In case (ii), $\triangle AOB$ contains K. In case (iii), the two sides adjacent to $\overline{AB}$ are parallel. We shall show that in each case it is possible to construct a polygon of area less than P containing K; this contradiction will then show that the midpoints of each side of the minimal n-gon must indeed meet K.

In case (i), since M is exterior to K, there exists a supporting line l of K separating M from K. (A supporting line of K is a line L such that K lies on one side of L and $K\cap L\neq\varnothing$.) Let m be the line through M parallel to l (Figure VI, (i)). Note that m cuts off a triangle of area strictly larger than area $(\triangle AOB)$, by the last lemma. But l cuts off an even larger triangle; hence, using l and P we can produce an n-gon containing K and having smaller area than P.

In case (ii), we can use the property (c) of the lemma to produce a line segment $\overline{CD}$ with its midpoint Q on $\overline{AB}$, $Q\neq M$, and $\overline{CD}$ not intersecting K (as depicted in Figure VI, (ii)). By the lemma again, $\triangle COD$ has strictly smaller area than all other triangles cut off by lines through Q; hence, area $(\triangle COD)<$area $(\triangle AOB)$. It then is clear that we can produce an n-gon of smaller area than P containing K.

In case (iii), l is chosen to be a supporting line of K at the point (nearer M) where the midline (of the two parallel sides) intersects the boundary of K. Then m is the line through M parallel to l. It is easy to see that using m and P we obtain an n-gon containing K with the same area as P, and using l we obtain an n-gon of smaller area. This completes the proof.

5. The smallest triangle containing a convex region. We now answer a question raised at the end of Section 2.

THEOREM 5. *Every convex region K is contained in some triangle of at most twice its area.*

Proof. It suffices to prove that if T_0 is a triangle of minimum area containing K, then area $(T_0) \leqq 2$ area (K). In order to do this, let T_0 be a minimal triangle circumscribed about K, with the midpoints A, B, C of its sides on K. As indicated in Figure VII, let T be the triangle similar to T_0 formed by drawing supporting lines of K parallel to the sides of T_0, and let A', B', C' be points where the sides of T meet K.

We shall now show that the hexagon $AB'CA'BC'$ has area at least twice the area of $\triangle ABC$. It will then follow that

$$\text{area } (K) \geqq \text{area } (AB'CA'BC') \geqq 2 \text{ area } (\triangle ABC) = \tfrac{1}{2} \text{ area } (T_0),$$

and this will prove the theorem.

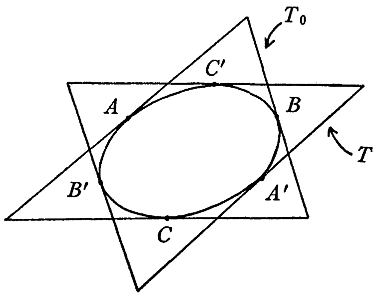

FIG. VII.

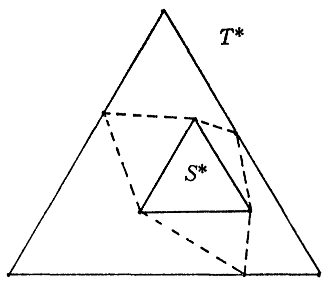

FIG. VIII.

We now affinely transform the configuration in Figure VII so that T is mapped onto an equilateral triangle T^*. Then $\triangle ABC$ is mapped onto an equilateral triangle S^* inside T^* with sides parallel to those of T^*, and the hexagon $AB'CA'BC'$ is mapped to a hexagon like that dotted in Figure VIII.

Since T_0 is minimal and similar to T, each side of T is at least as long as the corresponding side of T_0, hence, at least twice as long as the parallel side of $\triangle ABC$. Thus, the sides of T^* are at least twice the length of the sides of S^*. Since affine transformations preserve ratios of areas, we need only show that under these conditions the dotted hexagon in Figure VIII has at least twice the area of S^*. In order to show this, form the dotted hexagon in Figure IX by dropping perpendiculars to the sides of T^* from the centroid O of S^*.

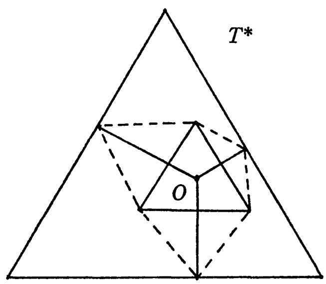

FIG. IX.

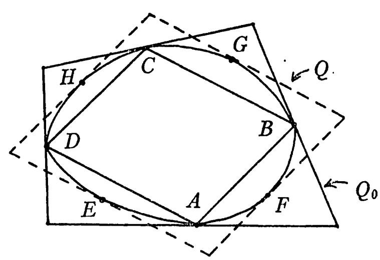

FIG. X.

The dotted hexagon in Figure IX has the same area as that in Figure VIII. We now leave it as an exercise to the reader to prove that this hexagon has at least twice the area of the small triangle S^*. [Hint: the sum of the lengths of the perpendiculars dropped to the sides of an equilateral triangle from any interior point is always equal to the altitude of the triangle. Keep in mind that the large triangle has sides at least twice the length of those of the small triangle.] With this, the reader completes the proof.

REMARKS. Theorem 5 was first proved by Gross [5]. Gross also proves that if the minimal triangle containing K has exactly twice the area of K, then K must be a parallelogram. This answers affirmatively our question (a) raised at the end of Section 2.

It is natural to ask for analogues of Theorem 5 for n-gons with $n>3$. The following partial result, pertaining to the case $n=4$, appears to be new:

THEOREM 6. *Every convex region K is contained in a quadrilateral Q_0 such that area $(Q_0) \leqq (\sqrt{2})$ area (K).*

Proof. Let Q_0 be a quadrilateral of minimum area containing K, with the midpoints A, B, C, D of its sides on K. As is well known, $ABCD$ is a parallelogram with area half that of Q_0. In Figure X we have drawn with dotted lines the parallelogram Q circumscribed about K with sides parallel to those of $ABCD$ and meeting K in points $EFGH$.

If we could show that the octagon Z with vertices $AFBGCHDE$ satisfies

$$\text{area } (Z) \geqq \sqrt{2} \text{ area } (ABCD),$$

then we would have

$$\text{area } (K) \geqq \text{area } (Z) \geqq \sqrt{2} \text{ area } (ABCD) = \frac{\sqrt{2}}{2} \text{ area } (Q_0),$$

and the theorem would follow.

We note that the area of Z is unchanged if we let E, F, G, and H move on their respective sides of Q, and, moreover, it suffices to consider the case where Q and $ABCD$ are rectangles (using an appropriate affine transformation). In other words, it suffices to show that area $(Z) \geqq \sqrt{2}$ area $(ABCD)$ in a situation like that depicted in Figure XI.

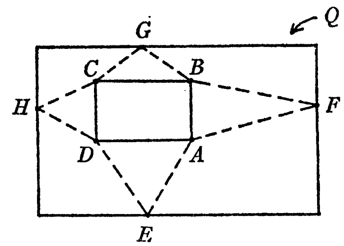

FIG. XI.

In Figure XI, E, F, G, and H are the feet of the perpendiculars dropped from the center of $ABCD$ into the sides of Q. The dotted polygon is our new Z. If

s and t are the lengths of the sides of $ABCD$, and s' and t' the lengths of corresponding parallel sides of Q, then it is easily checked that

$$\text{area }(Z) = \tfrac{1}{2}(st' + s't).$$

Recalling that area $(Q) \geqq$ area $(Q_0) = 2$ area $(ABCD)$ in our original configuration, we see that $s't' \geqq 2st$. Thus, using the fact that the arithmetic mean of two numbers is always at least their geometric mean, we obtain

$$\text{area }(Z) = \tfrac{1}{2}(st' + s't) \geqq \sqrt{st's't} \geqq (\sqrt{2})st = \sqrt{2}\text{ area }(ABCD),$$

and the theorem follows.

Remark. We do not know if Theorem 6 is a best possible result. It is possible that every K is contained in some quadrilateral Q such that area $(Q) \leqq \lambda$ area (K), where $\lambda < \sqrt{2}$. The question which needs to be answered is the following: *Is there a convex region K all of whose circumscribed quadrilaterals have area at least $(\sqrt{2})$ area (K)?*

Fejes Tóth [3, p. 38] remarks that the answers to corresponding questions for n-gons, $n>3$, are unknown.

Theorem 6 is of interest in connection with problems of "packing" convex sets. A distribution of nonoverlapping congruent copies of K in the plane is called a packing. The basic question is: how large a fraction of the plane can be covered by nonoverlapping copies of K? In other words, what is the highest "density" of packing which can be achieved? If Q_0 is the minimal quadrilateral containing K, it is possible to cover the plane with nonoverlapping copies of Q_0. Then we obtain a packing with copies of K having density $\geqq \sqrt{2}/2 > .707$. This packing even has a certain amount of regularity. It is the union of two "lattice" packings by K. The reference [3] contains a great deal of valuable information about packing problems.

6. Some familiar extremum problems. Another standard exercise in calculus texts is the following:

Given an ellipse E with semiaxes a and b, find the rectangle R_0 of maximum area inscribed in E.

In solving this problem, it is usually assumed that the sides of the rectangle are parallel to the axes of the ellipse. In order to justify this assumption, one needs to know that any rectangle R inscribed in E has its sides parallel to the axes (we are assuming E is not a circle). Let us show how to prove this fact using affine transformations.

Assume R is a rectangle inscribed in E. Affinely transform E to a circle E^*. Then, under the same transformation, R is sent to a parallelogram R^* inscribed in E^*. Now it is a trivial exercise to show that any parallelogram inscribed in a circle must be a rectangle. But the fact which interests us is that the center of R^* coincides with the center of E^*; hence, the center of R coincides with the center of E. Thus, the circumscribed circle C of R is centered at the center of E. Now it is obvious that such a circle C intersects E in four points which are the vertices of a rectangle with sides parallel to the axes; hence, R is such a rectangle.

Exercise 4. Use affine transformations to reduce the problem of finding R_0 to the problem of finding the maximum rectangle inscribed in a circle.

The result in the following exercise, intimately related to Exercise 2, can be established readily with an affine transformation.

Exercise 5. Prove that the maximum area of any triangle inside an ellipse E is $(3\sqrt{3}/4\pi)$ area (E).

The following theorem, proved in [3, p. 36] is complementary to our considerations concerning circumscribed n-gons of minimum area:

THEOREM 7. *If P is an inscribed n-gon of maximum area in a convex region K, then*

$$\text{area } (P) \geqq \frac{n}{2\pi} \sin \frac{2\pi}{n} \text{ area } (K),$$

and equality holds only if K is an ellipse.

REMARKS. Some of the examples of this article are also discussed in the paper of Klamkin and Newman [6]. The following interesting exercise is given there:

Exercise 6. Through a given point inside an ellipse, draw a line cutting off minimum area.

Although other examples, more or less familiar, do exist, we stop at this point.

References

1. M. T. Bird, Maximum rectangle inscribed in a triangle, to appear.
2. L. Danzer, D. Laugwitz, and H. Lenz, Über das Löwnersche Ellipsoid und sein Analogen unter den einem Eikörper einbeschriebenen Ellipsoiden, Arch. Math., 8 (1957) 214–219.
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Addendum to the 1970 Index

On pages 292 and 294 of the November–December 1970 issue of this MAGAZINE, under ARTICLES BY TITLE and ARTICLES BY AUTHOR, entries covering the following should be added:

Lindstrom, P. A., Evaluation of double integrals by means of the definition 85

On page 296, under SOLUTIONS, the following should be added:

Farrand, Rosalie, 104

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THE ALGEBRA OF REFLEXIVE RELATIONS

F. D. PARKER, St. Lawrence University

Introduction. One of the important concepts in algebra is that of a ring; that is, a set of elements R with two well-defined operations, addition and multiplication, such that (1) under addition, R is an abelian group, (2) under multiplication, R is associative (a semigroup) and (3) multiplication is both left and right distributive over addition. The fact that there are so many examples of rings (e.g., the integers, polynomials with integral coefficients, matrices) makes the abstract theory of rings an efficient way of studying a large class of mathematical systems. For more specialized systems, a ring may be further restricted (e.g., commutative rings, rings with an identity, euclidean rings, principal ideal rings, division rings, fields, etc.)

Yet there are interesting systems with two operations which do not meet all the requirements of a ring, such as the positive integers, polynomials with positive integral coefficients, matrices with nonnegative entries. Such systems have enough interesting properties and structure theory to make their study attractive.

At the outset we are embarrassed with the richness of possibilities. We can relax such restrictions as associativity, one or both of the distributive laws, or the fact that under addition we have an abelian group. In this paper, we study some of the properties of reflexive square Boolean matrices under addition and multiplication. A natural ordering is defined and an ideal theory is developed which has some interesting applications.

Reflexive Boolean matrices. We begin by considering the set S of square reflexive Boolean matrices of a fixed order n. The adjective Boolean means that the entries are either 0 or 1 and that the usual laws of Boolean algebra hold ($0+0=0$, $1+0=1+1=1$, $0\cdot 0=0\cdot 1=1\cdot 0=0$, and $1\cdot 1=1$). The adjective reflexive means that the entries on the main diagonal are all 1. Thus we have a finite set S of $2^{n(n-1)}$ elements.

With the usual laws of matrix addition and multiplication we easily see that addition is commutative and associative (that is, $\{S, +\}$ is an abelian semigroup), that $\{S, \cdot\}$ is a semigroup, and that multiplication is both right and left distributive over addition.

Since every relation on a finite set can be described by a Boolean matrix, and since the algebra of relations coincides with the addition and multiplication of these matrices [1], then one obvious application is to the algebra of relations. Another application is in finite graph theory, where the powers of the matrix describing the graph give information about the connectivity of the graph. A third application to logic has been described in [2]. More recently, it has been shown [3] that every topology on a finite set can be described by a square reflexive matrix M such that $M^2=M$. Although it is not explicitly stated in [3], nevertheless it is easy to see that every square reflexive Boolean matrix with this property corresponds to a partial ordering (a reflexive and transitive relation) of a set of n elements.

Some properties of $\{S, +, \cdot\}$ can be quickly ascertained, and others have been developed in [2]. Although we will turn to an axiomatic development in the next section, we mention here some of these properties.

1. Under addition, every element is an idempotent, that is, $x+x=x$ for all $x \in S$.
2. For all $x \in S$, $x^{n+1}=x^n$.
3. The identity matrix I has the property that $x+I=I+x=x$, and $xI=Ix=x$ for all $x \in S$.
4. The matrix u whose entries are all 1 has the property that $x+u=u+x=ux=xu=u$ for all $x \in S$.
5. We define two matrices x and y as equivalent if $x^n=y^n$. This is an equivalence relation, each equivalence class corresponds to a partial ordering (and a finite topology) of a set of n elements, and the elements of any equivalence class are closed under addition and multiplication.
6. It is natural to define $x \leqq y$ if matrix y has the entry 1 when the corresponding entry of x is 1. Then $\leqq$ is a partial ordering, and, for any matrices $x, y \in S$, we have $I \leqq x \leqq x+y \leqq xy \leqq u$.

Although more observations can be made, we turn now to an axiomatic treatment which will eventually yield all of these properties.

An axiomatic treatment. Let S be a finite set with the two well-defined operations of addition and multiplication with the following properties:

1. Under addition, S is an abelian semigroup.
2. Under multiplication, S is a semigroup.
3. Multiplication is right and left distributive over addition.
4. There is an element α which serves as a right identity for *both* addition and multiplication, and there is an element β which serves as a left identity for both addition and multiplication.
5. There is an element ρ which serves as a right *zero* for both addition and multiplication, and there is an element σ which serves as a left zero for both addition and multiplication. (To say that ρ is a right zero for multiplication means that $s\rho=\rho$ for all $s \in S$.) Then S is a *Boolean semiring* (bsr).

THEOREM I. *For any element $x \in S$, $x+x=x$.*

Proof. $x+x=x\alpha+x\alpha=x(\alpha+\alpha)=x\alpha=x$. In terms of [**4**], S, is a *commutative band* under addition.

THEOREM II. *S has only one identity, which serves as both a right and left identity. S has only one zero, which is both a right and left zero.*

The proof may be found in [**4**]. We will from now on denote the identity by I, the zero by 0.

DEFINITION. *We define $x \leqq y$ or $y \geqq x$ to mean $x+y=x$.*

A word of explanation is helpful here. In the example of the Boolean reflexive matrices, the zero element is the matrix all of whose entries are 1, the identity is the usual identity matrix. Since the zero element has more 1's than the identity, we might have defined $x \leqq y$ to mean $x+y=y$. Nevertheless, we prefer the definition as it stands above. In this way we will agree with [4] and at the same time have $0 \leqq I$, rather than $I \leqq 0$.

THEOREM III. *The relation* $\leqq$ *is a partial ordering.*

Proof. We need to show (1) if $x \leqq y$ and $y \leqq x$, then $x=y$ and (2) if $x \leqq y$ and $y \leqq z$, then $x \leqq z$.

(1) If $x \leqq y$, then $x+y=x$. If $y \leqq x$, then $y+x=x+y=y$, and therefore $x=y$.

(2) If $x \leqq y$, then $x+y=x$. If $y \leqq z$, then $y+z=y$. Then $x+z=(x+y)+z=x+(y+z)=x+y=x$, and $x \leqq z$.

THEOREM IV. *For any elements* x, y, *of* S, *we have* $0 \leqq xy \leqq x+y \leqq x \leqq I$.

Proof. Since $xy+0=0$, then $0 \leqq xy$. Since $xy+x+y=xy+x+xy+y=x(y+I)+(x+I)y=xy+xy=xy$, then $xy \leqq x+y$. Since $x+y+x=x+y$, then $x+y \leqq x$. Since $x+I=x$, then $x \leqq I$.

THEOREM V. *If* $x \leqq y$, *then* $xz \leq yz$, *and* $zx \leqq zy$.

Proof. If $x \leqq y$, then $xz+yz=(x+y)z=xz$, and hence $xz \leqq yz$. Similarly $zx+zy=z(x+y)=zx$, hence $zx \leqq zy$.

THEOREM VI. *For any* $x, y, z \in S$, $xzy \leqq xy$.

Proof. $xzy+xy=xzy+xIy=x(z+I)y=xzy$.

Just as any element of a group generates a cyclic subgroup, any element of a semigroup also generates a subsemigroup. In [4] it is shown that the set $\langle x\rangle=\{x, x^2, x^3, \cdots\}$ either has all elements different, in which case $\langle x\rangle$ is isomorphic to the semigroup of natural numbers, or else there is a first repetition, say $x^r=x^s$, $r<s$. In this case not only is $\langle x\rangle$ a subsemigroup, but the elements $\{x^r, x^{r+1}, \cdots, z^{s-1}\}$ form a cyclic group. In the case of a bsr, we have the following interesting result:

THEOREM VII. *There is an integer* r *associated with a* bsr S *such that* $x^r=x^{r+s}$ *for* $s \geqq 1$ *for any element* $x \in S$.

Proof. Since S is finite, there is a first repetition, say $x^k=x^{k+s}$. From Theorem IV, $x^{k+s} \leqq x^{k+1} \leqq x^k$. If $x^{k+s}=x^k$, then $x^k \leqq x^{k+s} \leqq x^{k+1}$, and hence $x^k \leqq x^{k+1}$, thus $x^k=x^{k+1}$, and $x^k=x^{k+1}=x^{k+2}= \cdots$. Since there is such a positive integer k for every $x \in S$, let r be the maximum value of k for all $x \in S$.

The importance of this result lies not only in the results which follow, but in some practical applications. In finite graph theory, the connectivity of a graph of n points is found from the matrix $M+M^2+\cdots+M^n$, where M is the Boolean matrix corresponding to the graph. If the graph has n loops the connectivity is found more directly from M^n. In [2], theorems which can be proved by one law of logic from a set of postulates can be found by raising a Boolean matrix to a given power, and higher powers do not add new theorems. In stochastic matrices, the question of whether or not a given matrix is regular can be decided by considering powers of a Boolean matrix. A careful reading of

[3] will reveal an application to finite topologies. Consider a subset of the power set of $\{x_1, x_2, \cdots, x_n\}$. If we let S_i be the intersection of the sets containing x_i, $i=1, 2, 3, \cdots, n$, and build a reflexive Boolean matrix M from $\{S_i\}$, then the condition for a finite topology is that $M^2=M$. If $M^2\neq M$, then M^{n-1} describes a minimum topology from a set of sets $\{T_i\}$ where $T_i\supset S_i$, $i=1, 2, \cdots, n$; that is, minimum in the sense that $\{T_i\}$ is the minimum set containing the set S_i such that $\{T_i\}$ forms a basis for the topology.

DEFINITION. *If $x^r=y^r$, then we say that x is equivalent to y ($x\sim y$).*

Clearly this relation is reflexive, symmetric and transitive, so that this is an equivalence relation, thus partitioning the elements of S.

The number of classes of $S/\sim$ has an interesting application. If S is isomorphic to the set of reflexive Boolean matrices of order n, then the number of equivalence classes is at the same time equal to the number of topologies on a set of n elements and also the number of partial orderings of the same set.

THEOREM VIII. *If $x\sim y$, then $x+y\sim x$ and $xy\sim x$.*

Proof. $(xy)^r=(xy)(xy)\cdots(xy)\geqq(xy^r)(xy^r)\cdots(xy^r)=(xx^r)(xx^r)\cdots(xx^r)=x^{r^2+r}=x^r$. But since $xy\leqq x$, $(xy)^r\leqq x^r$. Hence $(xy)^r=x^r$ and $xy\sim x$. Since $x+y\leqq x$, then $(x+y)^r\leqq x^r$. Since $x\geqq x^r$, $y\geqq y^r$, then $(x+y)\geqq x^r+y^r=x^r+x^r=x^r$. Then $(x+y)^r\geqq x^{r^2}=x^r$, and hence $(x+y)^r=x^r$, and $x+y\sim x$.

THEOREM IX. *If $x\geqq y\geqq x^r$, then $y\sim x$.*

Proof. Since $y\geqq x^r$, then $y^r\geqq x^{r^2}=x^r$. Since $x\geqq y$, then $x^r\geqq y^r$, so that $x^r=y^r$ and $x\sim y$.

In the study of algebraic systems, we are often concerned with subsystems, and our last few theorems have been preliminary to describing some of these sub-bsr's.

THEOREM X. *Let x be any element of S, and let $E(x)$ be all the elements of S which are equivalent to x. Then $E(x)\cup I$ is a* bsr.

Proof. By Theorem VIII, $E(x)$ is closed under addition and multiplication, and it remains closed under the adjunction of I. The element I serves as the identity and x^r serves as the zero element. To see this let $y\in E(x)\cup I$. If $y=I$, then $yx^r=x^ry=x^r$. If $y\neq I$, then $yx^r\geqq y^rx^r=x^rx^r=x^r$. But $yx^r\leqq x^r$, so that $yx^r=x^r$. In the same way we can show that $x^ry=x^r$.

THEOREM XI. *Let x be any element of S. Then $\langle x\rangle\cup I$ is a* bsr.

Proof. The proof is almost immediate. The element I is the identity, x^r is the zero. The set $\langle x\rangle\cup I$ is closed under addition, since $x^p+x^q=x^t$, where $t=\min(p, q)$.

In semigroup theory, ideals play an important role, so we turn to the ideals of a bsr.

DEFINITION. *Let x be any element of S. The set Sx (the totality of elements sx, $s\in S$) is the left multiplicative ideal generated by x, and xS is the right multiplicative ideal generated by x. The set $S+x=x+S$ is the additive ideal generated by x.*

THEOREM XII. *The ideals Sx, xS and $S+x$ are all closed under addition and multiplication.*

Proof. Since $s_1x+s_2x=(s_1+s_2)x=s_3x$, Sx is closed under addition. Since $s_1xs_2x=s_3x$, Sx is closed under multiplication. Clearly, $S+x$ is closed under addition. It is easy to show that $S+x$ consists of all (and only those) elements which are $\leqq x$. Then we have $(x+s_1)(x+s_2)\leqq x^2\leqq x$, so that $S+x$ is closed under multiplication.

THEOREM XIII. *Let x be any element of S. Then $Sx\cup I$, $xS\cup I$, and $S+x$ are all sub*-bsr's.

Proof. Having established closure, we need only to show the existence of a zero and an identity. The zero element 0 is in each of these ideals, and the element x serves as the identity for $S+x$

THEOREM XIV. *The ideals xS and Sx contain only elements y such that $y\leqq x$. The ideal $S+x$ contains all elements z such that $z\leqq x$. The ideal $S+x$ is a lower semilattice with respect to $\leqq$.*

Proof. The proof of the first two statements is immediate; the proof of the third statement can be found in [**4**], page 24.

THEOREM XV. *Let x be a fixed element of S and $C(x)$ those elements which commute with x. Then $C(x)$ is a sub*-bsr.

Proof. Since I and 0 are elements of $C(x)$, we need only show closure. Let y_1 and y_2 be elements of $C(x)$. Since $x(y_1+y_2)=xy_1+xy_2=y_1x+y_2x=(y_1+y_2)x$, $C(x)$ is closed under addition. Since $xy_1y_2=y_1xy_2=y_1y_2x$, $C(x)$ is closed under multiplication.

Theorems X through XV give us several ways of forming sub-bsr's from a fixed element x, not necessarily all different. The sets $E(x)\cup I$, $\langle x\rangle\cup I$, $Sx\cup I$, $xS\cup I$, $S+x$, and $C(x)$ are all sub-bsr's. Moreover, any element x such that $x^2=x$ is also a sub-bsr. We turn now to a consideration of homomorphisms.

DEFINITION. *Let S be a* bsr *and T a set with two well defined operations. If there is a mapping f of S into T such that $f(x+y)=f(x)+f(y)$ and $f(xy)=f(x)f(y)$, then we say that S is homomorphic to T, and this is denoted by $S\sim T$.*

THEOREM XVI. *If S is a* bsr *and $S\sim T$, then T is a* bsr. *The mapping preserves the natural ordering.*

Proof. It is easy to see that the identity of T is $f(I)$ and the zero of T is $f(0)$. If $x\leqq y$, then $x+y=x$, $f(x+y)=f(x)+f(y)=f(x)$, so $f(x)\leqq f(y)$.

In the theory of groups, the kernel of a homomorphic mapping (those elements which map into the identity) form a subgroup. In our theory the corresponding result is not quite as nice, as evidenced by our next theorem.

THEOREM XVII. *Let S be a* bsr *and $S\sim T$. Denote by K_I those elements of S which map into the identity I' of T and by K_0 those elements of S which map into the zero of T. Then $K_I\cup 0$ is a* bsr *and so is $K_0\cup I$.*

Proof. Let k_1, k_2 be elements of K_I. Then $f(k_1+k_2)=f(k_1)+f(k_2)=I'+I'=I'$. Also $f(k_1k_2)=f(k_1)f(k_2)=I'I'=I'$. Therefore K_I is closed under both operations. Since $f(x)=f(x+I)=f(x)+f(I)$, then $f(I)=I'$. The adjunction of 0 to K_I provides the zero which makes $K_I\cup 0$ a bsr. The proof for $K_0\cup I$ is similar.

Questions and conclusions. Some important questions have been ignored in this short discussion, questions which can provide the basis of further investigation.

Are the postulates independent? A few examples will give a partial answer.

The set K_I in Theorem XVII satisfies all the postulates except Postulate 5, and the set K_0 in the same theorem satisfies all but Postulate 4. The set $\{0, x, 1\}$ with the commutative addition laws such that 0 is a zero, 1 is an identity and $x+x=x$, and with the same multiplication table with the exception that $xx=0$ satisfies all the postulates except Postulate 3. In this system $x(1+x)=0$, but $x\cdot 1+x\cdot x=x$.

Are there bsr's of any given order? The answer is yes. Certainly there is a bsr of order one. Furthermore if there exists a bsr S of order k, a new element z can be added such that $z+x=z$, $zx=xz=z$. Then z is the zero, all the requirements are met and the order of the new bsr S' is $k+1$. The zero of S is no longer the zero of S'.

Are any two bsr's of the same order isomorphic? Here the answer is negative. Consider the noncommutative bsr of the reflexive Boolean matrices of order three, whose order is 64. On the other hand, we can start with a single element, then successively add new zeros until the order is 64. This new bsr is commutative. In fact it is isomorphic to the set of integers $\{1, 2, 3, \cdots, 64\}$ in which the operation tables for addition and multiplication are *identical*: specifically $x+y=xy=\max\{x, y\}$. In this case the ordering is a linear ordering.

Consider any finite set T with a linear ordering. Let $T=\{x_1, x_2, \cdots x_n\}$ be such a set in which $x_i\leqq x_j$ if $i\leqq j$. If we define $x_i+x_j=x_ix_j=x_k$ where $k=\min(i, j)$, then all the requirements of a bsr are satisfied; x_1 is the zero and x_n is the identity.

If the two operations have the same Cayley table (i.e., $x+y=xy$), is the partial ordering a linear ordering? The answer is negative, as shown by the counterexample in which the Cayley table is

1	x	y	0
x	x	1	0
y	1	y	0
0	0	0	0

The abundance of bsr's is shown by the fact that every finite semigroup generates a bsr [5]. Consider any semigroup S. If S has no identity, we adjoin an identity I in the usual manner; and henceforth refer to the (new) semigroup as S. Form the collection $\mathcal{S}$ of all the subsets of S which contain I and label them $\mathcal{S}=\{I, A, B, C, \cdots S\}$. Define AB to be the collection of all elements ab such that $a\in A$, $b\in B$. Then $AB\in\mathcal{S}$, and $(AB)C=A(BC)$, so that $\mathcal{S}$ forms a semigroup. Moreover $AS=SA=S$ and $AI=IA=A$. We define $A+B$ to be the *union* of the two sets A and B. Then $A+B=B+A$, $A+(B+C)=(A+B)+C$, $A+I=A$, and $A+S=S$. In addition multiplication is distributive over addition, so $\mathcal{S}$ forms a bsr.

Moreover, any bsr formed in this way is isomorphic to a set of reflexive Boolean matrices. Consider the semigroup $S=\{1, 2, 3, \cdots n\}$ where 1 is the identity of S. We form a matrix $M=m_{ij}$ corresponding to the element k in the following way; if $ik=j$, then $m_{ij}=1$, and all other entries in row i are zero. In this way we form a matrix M_k whose row-sums are all 1. To every element of S there corresponds such a matrix, and the existence of an identity in S guarantees that each matrix is different, thus establishing a 1-1 correspondence. Moreover, the identity of S corresponds to the identity matrix I. Multiplication is preserved, thus giving an isomorphism between S and S', a set of matrices. Now if we consider $\mathcal{S}=\{I, A, B, \cdots S\}$ as all the subsets of S' containing I, we see that a bsr is formed; if $A=\{I, M_1, M_2, \cdots M_r\}$, and $B=\{I, M_1', M_2', \cdots M_s'\}$, then $A+B=\sum_{i=1}^r M_i+\sum_{i=1}^s M_i'+I$, where the addition on the right hand side is the Boolean sum of the matrices, and $AB=(\sum_{i=1}^r M_i+I)(\sum_{i=1}^s M_i+I)$, where the multiplication on the right hand side is the Boolean product of the matrices.

As one example, the semigroup whose multiplication table is

	a	*b*	*c*	*d*	*e*	*i*
a	*a*	*a*	*a*	*d*	*d*	*a*
b	*a*	*b*	*c*	*d*	*d*	*b*
c	*a*	*c*	*b*	*d*	*d*	*c*
d	*d*	*d*	*d*	*a*	*a*	*d*
e	*d*	*d*	*d*	*a*	*a*	*e*
i	*a*	*b*	*c*	*d*	*e*	*i*

generates a bsr whose zero element is

$$\begin{matrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}$$

Can any bsr be represented by a set of reflexive Boolean matrices? While the answer is not known, the bsr of order 64 with the linear ordering can be represented by matrices. Simply take the reflexive matrices of order 64, and let the integer p correspond to the matrix which has the first p elements of the first row equal to 1, and all other off-diagonal entries are 0.

There are other questions not answered here:

(1) Given two integers $p<n$, can a bsr of order n with exactly p equivalence classes be formed?

(2) There is another natural operation in Boolean matrices, in which the "product" of two matrices is found by multiplying corresponding *elements* (Hadamard product). In this case, the identity serves as a zero under this new operation. This new operation is commutative, associative, distributive over addition, and addition is distributive over it. Investigation analogous to that already carried out seems promising.

(3) Given a set T of n elements how many partial orderings are there on T? The question is related, as we have seen, to finite topologies; whether an axiomatic treatment of bsr's is of any help is not clear.

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2. F. D. Parker, Boolean matrices and logic, this MAGAZINE, 37 (1964) 33–38.
3. V. Krishnamurthy, On the number of topologies on a finite set, Amer. Math. Monthly, 73 (1966) 358–361.
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5. R. R. Stoll, Representations of finite simple semigroups, Duke Math J., 11 (1944) 475–481.

FACTORING FUNCTIONS

J. C. BODENRADER, State University College of Arts
and Science, Plattsburgh, New York

1. Introduction. A most useful result in number theory is the one which states that each composite natural number may be written as a product of primes in one and only one way. The result actually consists of two parts: the first asserts the existence of the desired type of factorization and the second asserts that the factorization is unique. The purpose of the present article is to publicize a similar result which holds in the case of functions between sets. In a leisurely way the article shows how each function between sets may be factored (in the sense of composition) into a triple product involving first an onto, then a 1-1 and onto, and finally a 1-1 function. The question of uniqueness is discussed along with several important applications including applications to finite mathematics and to calculus. Except for the applications it is assumed only that the reader is familiar with the notions of 1-1 and onto as they apply to functions and that at one time or another he has had occasion to deal with commutative diagrams of functions.

2. The problem. Suppose for the moment that A and B are sets and that f is a function from A to B. We know only that A and B are sets and that f is a function from A to B. No other particulars are given. In this general setting the problem is to show that there exists an onto function P_f, a 1-1 and onto function $\bar{f}$, and a 1-1 function I_f, such that $f = I_f \circ \bar{f} \circ P_f$. As is quite often the case in mathematics we begin by considering a special case in a way which will easily lend itself to generalization later on.

(2) There is another natural operation in Boolean matrices, in which the "product" of two matrices is found by multiplying corresponding *elements* (Hadamard product). In this case, the identity serves as a zero under this new operation. This new operation is commutative, associative, distributive over addition, and addition is distributive over it. Investigation analogous to that already carried out seems promising.

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3. The special case. Let A be the set $\{1, 2, 3, 4, 5\}$ and let B be the set $\{a, b, c\}$. Let f be the function from A to B which sends 1 to a, 2 to a, 3 to a, 4 to c, and 5 to c.

Call a nonempty subset of A a *component for f* if there is an element y in B such that the subset consists precisely of all elements of A which are sent by f to y. The set $\{1, 2, 3\}$ is then a component since it consists precisely of all elements of A which are sent by f to the element a in B. Clearly we obtain only two components, $\{1, 2, 3\}$ and $\{4, 5\}$. Let C_f denote the set of components, $C_f = \{\{1,2,3\},\{4,5\}\}$.

Let P_f be the function from A to C_f which sends each element of A to the unique component to which it belongs (for example, P_f sends 3 to $\{1, 2, 3\}$). Let R_f denote the range of f, $R_f = \{a, c\}$, and let $\bar{f}$ be the function from C_f to R_f which sends each component to the one element of R_f to which f sends every element of the component (for example, $\bar{f}$ sends $\{1, 2, 3\}$ to a). Finally, let I_f be the function from R_f to B which sends each element of R_f to itself. Then P_f is onto, $\bar{f}$ is 1-1 and onto, I_f is 1-1, and $f = I_f \circ \bar{f} \circ P_f$. The following commutative diagram provides a neat summary of our results thus far:

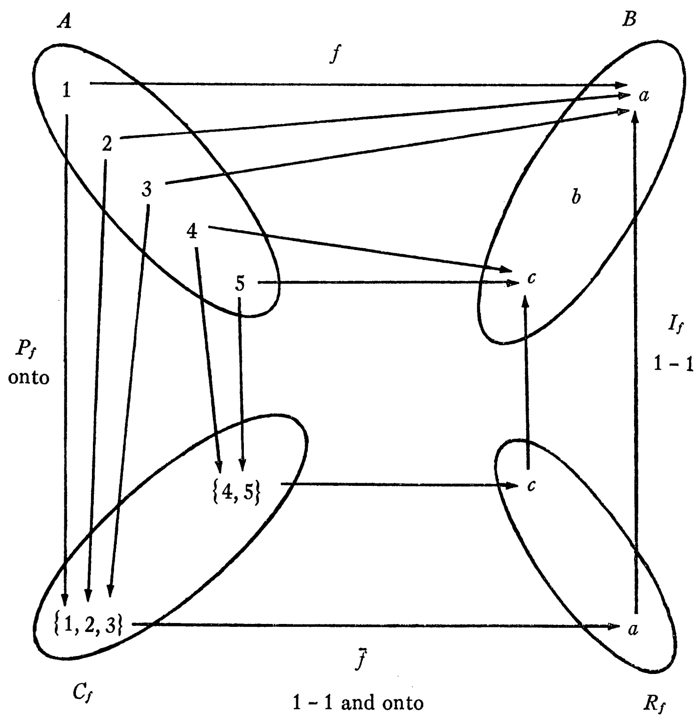

FIG. 1.

4. The factorization in general. Let A and B be sets and let f be a function from A to B. We will now generalize the argument given in Section 3 in order to obtain the desired factorization for f.

Each element of A determines a unique *component for f*. If x is an element of A and if f sends x to y then the component determined by x is the set consisting of all elements of A which are sent by f to y. If we denote the component determined by x as $[x]$ then $[x]=\{a\in A:f(a)=f(x)\}$. Let C_f denote the set of components and let P_f be the function from A to C_f which sends each element of A to its component, $P_f(x)=[x]$. Let R_f denote the range of f, $R_f=\{f(x):x\in A\}$, and let $\bar{f}$ be the function from C_f to R_f which sends each component to the one element of R_f to which f sends every element of the component, $\bar{f}([x])=f(x)$. Finally, let I_f be the function from R_f to B which sends each element of the range of f to itself, $I_f(y)=y$.

THEOREM 1. *P_f is onto, $\bar{f}$ is 1-1 and onto, I_f is 1-1, and $f=I_f \circ \bar{f} \circ P_f$.*

Proof. The proof is broken into several parts.

(1) P_f is onto. Each component is of the form $[x]$ for some element x in A and $P_f(x)=[x]$.

(2) $\bar{f}$ is 1-1. Let $[x_1]$ and $[x_2]$ be components and suppose $\bar{f}([x_1])=\bar{f}([x_2])$. Then $f(x_1)=f(x_2)$ so that x_1 and x_2 are sent by f to the same element in B. It follows that x_1 and x_2 belong to the same component and $[x_1]=[x_2]$.

(3) $\bar{f}$ is onto. Let y be an element of R_f. Then there is an element x in A such that $f(x)=y$. But then $\bar{f}([x])=f(x)=y$.

(4) I_f is 1-1.

(5) $f=I_f \circ \bar{f} \circ P_f$. Let x be an element of A. Then $I_f \circ \bar{f} \circ P_f(x)=I_f \circ \bar{f}([x]) =I_f(f(x))=f(x)$.

We summarize with a commutative diagram:

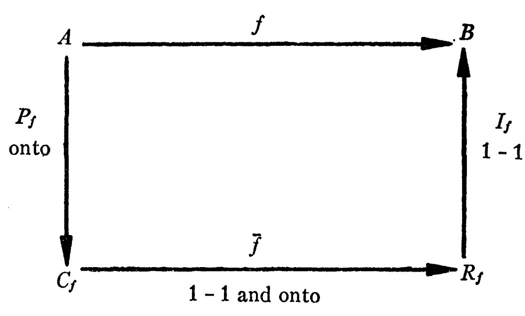

FIG. 2.

5. Uniqueness. The factorization for f given in Section 4 is unique in the following sense:

THEOREM 2. *Suppose there are sets C and R and functions $g_1\colon A\to C$, $g_2\colon C\to R$, and $g_3\colon R\to B$. Further, suppose that g_1 is onto, g_2 is 1-1 and onto, g_3 is 1-1, and $f=g_3 \circ g_2 \circ g_1$. Then there are functions $h_1\colon C_f\to C$ and $h_2\colon R\to R_f$ which are each 1-1 and onto and in addition the diagram in Figure 3 commutes.*

Proof. The functions h_1 and h_2 are defined as follows: $h_1([x])=g_1(x)$ for each component $[x]$ and $h_2(r)=g_3(r)$ for each element r in R. The remaining details of the proof are left for the reader.

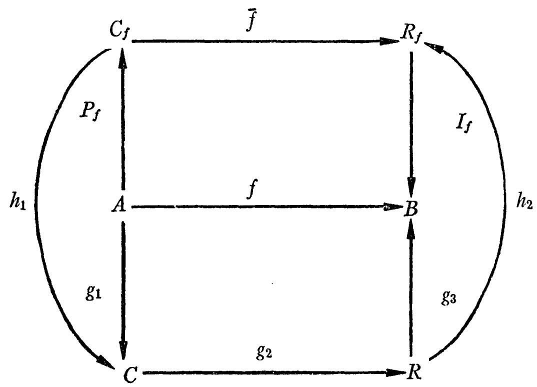

FIG. 3.

6. An application to finite mathematics. As a first application of our results let us seek to determine the number of different arrangements on all the letters of the word *LOLLY*. If we let B denote the set of arrangements of the type just mentioned then the problem is to determine the number of elements in B.

Let A denote the set of arrangements on the symbols $L_1OL_2L_3Y$ and let f be the function from A to B which "forgets" subscripts (for example, $f(L_1OL_3YL_2) = LOLYL$). As in our previous discussion let C_f and R_f denote respectively the set of components and the range of f.

Since f is onto, $R_f = B$. By Theorem 1 the function $\bar{f}$: $C_f \to B$ is 1-1 and onto. C_f and B therefore have the same number of elements. But A has $5! = 120$ elements and each component has $3! = 6$ elements. Therefore B has precisely $120/6 = 20$ elements.

7. An application to trigonometry. As a second application let us consider the trigonometric function cos. If R denotes the set of real numbers then cos is a function from R to R. Each real number determines a unique component for cos. The component determined by the real number x is the set $\{2n\pi \pm x : n = 0, \pm 1, \pm 2, \cdots\}$. Since each component contains exactly one real number between 0 and π we may identify the set of components $C_{\cos}$ with the set of real numbers in the closed interval $[0, \pi]$. The range of cos is the closed interval $[-1, 1]$ and applying Theorem 1 we obtain the commutative diagram:

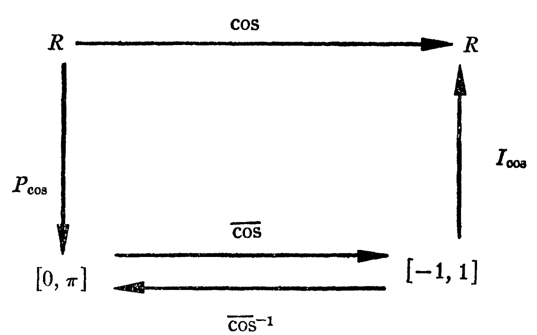

FIG. 4.

The function cos is neither 1-1 nor onto and so it does not have an inverse. However, since $\overline{\cos}$ is 1-1 and onto it does have an inverse which we have denoted as $\overline{\cos}^{-1}$. As displayed $\overline{\cos}^{-1}$ is actually the function "arc cosine." It sends each real number x in $[-1, 1]$ to the unique real number y in $[0, \pi]$ which satisfies the equation $\cos(y)=x$.

8. An application to calculus. The following situation is of considerable importance in calculus. Let $\mathfrak{R}$ denote the set of all functions from the reals to the reals and let $\mathfrak{D}$ denote the subset of $\mathfrak{R}$ consisting of the differentiable functions. Let D: $\mathfrak{D} \to \mathfrak{R}$ be the ordinary derivative operator which sends each differentiable function to its derivative. We then have the commutative diagram:

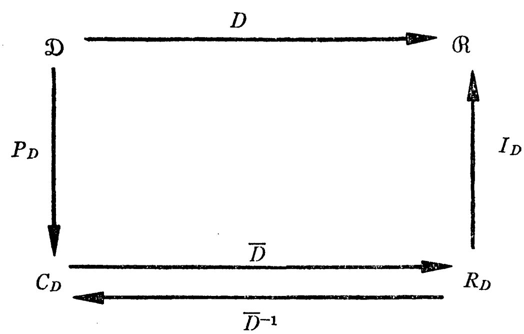

FIG. 5.

The component determined by an element f in $\mathfrak{D}$ is just the set $f+C$ consisting of f and all functions which differ from f by a constant function. Notice that the antiderivative operator appears here as $\overline{D}^{-1}$. It sends each antidifferentiable function to the unique component which is its antiderivative.

9. Other applications. When the sets A and B are endowed with structure and f is a structure-preserving function from A to B it is usually the case that C_f and R_f can be endowed with similar structure in such a way that Theorems 1 and 2 remain valid when appropriately restated in the setting of structured sets and structure-preserving functions. Some of the more important examples of this are the case for groups and homomorphisms, the case for vector spaces and linear transformations, and the case for topological spaces and continuous maps.

BUFFON IN THE ROUND

M. F. NEUTS and P. PURDUE, Purdue University

1. Introduction. In his paper *Essai d'arithmêtique morale* published in 1777, Buffon suggested what was essentially a new branch of probability-problems involving geometrical considerations. He proposed the following problem, which today is well known as *Buffon's needle problem:*

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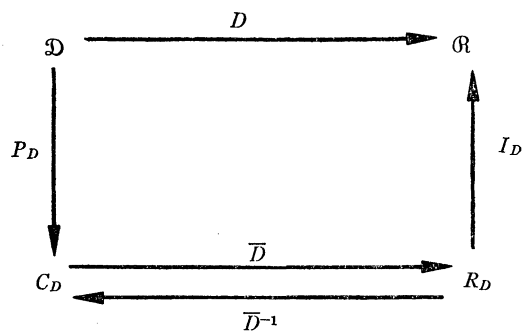

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"A floor is ruled with equidistant parallel lines $2a$ units apart; a rod of length $2c$, where $2c<2a$, is thrown at random on the floor. Find the probability of the rod intersecting one of the lines."

Buffon gave what is accepted as the correct probability, viz., $2c/\pi a$. In analyzing the solution we find that the essential points are these:

Let x be the distance from the center of the rod to one of the lines and let θ be the angle made by the rod with the direction of the lines. We then assume that x is uniformly distributed on $(0, a)$, that θ is uniformly distributed on $(-\pi/2, \pi/2)$ and that x and θ are stochastically independent; this is what Buffon means when he says that the rod is thrown "at random" onto the floor.

In 1812, in his book *Théorie analytique des Probabilités*, Laplace extended the original problem to a crisscross of two mutually perpendicular sets of parallel lines at distances a and b units apart. Laplace also used the above interpretation of the phrase "at random" and he found that for a rod of length l ($l<a$ or b) the probability of falling on one of the lines is $[2l(a+b)-l^2]/\pi ab$. We note that by letting either a or b go to infinity this reduces to Buffon's original result. For further information on the history of the problem we refer the reader to [**1**], [**2**], [**3**].

In this paper the authors discuss a lateral extension of Buffon's problem. Instead of parallel lines we suppose that a circle of radius R is drawn on the floor. A needle of length $2d$ is thrown at random onto the circle so that its midpoint falls within or on the circumference. Just as in the original problem, we must explain what we mean by the term "at random." Here we consider two different interpretations of the term. In one of these, which we will call case A, we suppose that prior to our throwing the needle, we designate a radius vector along which the center of the needle must lie. If we let U denote the distance from the center of the circle to the midpoint of the needle we then have that U is uniformly distributed in $(0, R)$. We also assume that θ, the angle made by the needle with a fixed vector, is uniformly distributed on $[0, \pi]$ and that U and θ are stochastically independent. In the second case, case B, we assume that the probability that the center of the needle falls in any subset of the circle is just the area of the subset divided by the total area of the circle. We again make the same assumptions about θ.

Formally then, we have the following:

A. Let U be the distance from the center of the circle to M. Then U has a density function given by:

$$f(u) = \begin{cases} 1/R & 0 \leqq u \leqq R \\ 0 & \text{otherwise.} \end{cases}$$

B. U has a density function given by:

$$g(u) = \begin{cases} 2u/R^2 & 0 \leqq u \leqq R \\ 0 & \text{otherwise.} \end{cases}$$

In both cases we assume that θ is uniformly distributed in $[0, \pi]$ and that U and θ are stochastically independent.

We shall find the probability distribution of the number of intersections of the needle with the circumference of the circle. In case A these probabilities are expressed in terms of elliptic integrals of the 1st and 2nd kinds. On the other hand, the results for case B are expressed in terms of elementary functions only. In Appendix 1 a table of these probabilities is given for certain d/R ratios.

2. Derivation of the probabilities. We define the random variable Z as the number of intersections of the needle with the circumference of the circle. It is clear, that for $d>2R$, $P[Z=2]=1$, so we need only consider the case $0<d\leqq 2R$. There are 2 main subcases to consider, (a) $R\leqq d\leqq 2R$ and (b) $0<d\leqq R$. We shall deal with case (a) first.

(a) $R\leqq d\leqq 2R$. We shall break this case down into two further subcases depending on the distance from 0 to M. Let $p_i(u)$ be the conditional probability $P[Z=i|\, U=u]$, $i=0, 1, 2$ then:

(i) $0<u\leqq d-R$. Under these assumptions we always have exactly two intersections, so that:

$$p_0(u) = p_1(u) = 0$$
$$p_2(u) = 1.$$

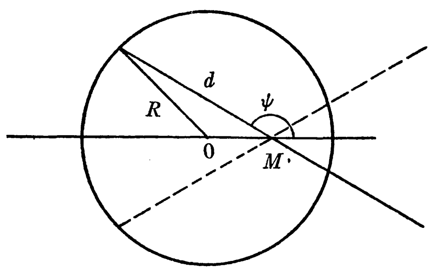


FIG. 1.

(ii) $d-R<u\leqq R$. Referring to Figure 1 for notation we have:

$$p_0(u) = 0$$
$$p_1(u) = 2 - 2\psi/\pi$$
$$p_2(u) = 2\psi/\pi - 1$$
$$R^2 = u^2 + d^2 - 2ud\cos(\pi - \psi).$$

On expressing ψ in terms of u, d and R and substituting we get

$$p_0(u) = 0$$
$$p_1(u) = 2 - 2/\pi \arccos\left(\frac{R^2 - d^2 - u^2}{2ud}\right)$$
$$p_2(u) = 2/\pi \arccos\left(\frac{R^2 - d^2 - u^2}{2ud}\right) - 1.$$

$P_A[Z=i]$ denotes the probability of i intersections in case A. Similarly for

$P_B[Z=i]$. It follows that for $R \leqq d \leqq 2R$ we have:

$$P_A[Z = 0] = 0$$

$$P_A[Z = 1] = \int_{d-R}^{R} \left[2 - \frac{2}{\pi} \arccos\left(\frac{R^2 - d^2 - u^2}{2ud}\right)\right]\frac{du}{R}$$

$$P_A[Z = 2] = \int_{0}^{d-R} \frac{du}{R} + \int_{d-R}^{R} \left[\frac{2}{\pi} \arccos\left(\frac{R^2 - d^2 - u^2}{2ud}\right) - 1\right]\frac{du}{R}\cdot$$

In case B we have

$$P_B[Z = 0] = 0$$

$$P_B[Z = 1] = \int_{d-R}^{R} \left[2 - \frac{2}{\pi} \arccos\left(\frac{R^2 - d^2 - u^2}{2ud}\right)\right]\frac{2udu}{R^2}$$

$$P_B[Z = 2] = \int_{0}^{d-R} \frac{2udu}{R^2} + \int_{d-R}^{R} \left[\frac{2}{\pi} \arccos\left(\frac{R^2 - d^2 - u^2}{2ud}\right) - 1\right]\frac{2udu}{R^2}\cdot$$

Simplifying these expressions we get

$$P_A[Z = 0] = 0,$$

$$P_A[Z = 1] = 4 - \frac{2d}{R} - \frac{2}{\pi R}\int_{d-R}^{R} \arccos\left(\frac{R^2 - d^2 - u^2}{2ud}\right) du,$$

$$P_A[Z = 2] = \frac{2d}{R} - 3 + \frac{2}{\pi R}\int_{d-R}^{R} \arccos\left(\frac{R^2 - d^2 - u^2}{2ud}\right) du,$$

$$P_B[Z = 0] = 0,$$

$$P_B[Z = 1] = \frac{2d}{R^2}(2R - d) - \frac{4}{\pi R^2}\int_{d-R}^{R} u \arccos\left(\frac{R^2 - d^2 - u^2}{2ud}\right) du$$

$$P_B[Z = 2] = 1 - \frac{2d}{R^2}(2R - d) + \frac{4}{\pi R^2}\int_{d-R}^{R} u \arccos\left(\frac{R^2 - d^2 - u^2}{2ud}\right) du.$$

We will leave the evaluation of the above integrals to Section 3.

(b) $0 < d \leqq R$. We break this case down into three subcases depending on the distance from 0 to M.

(i) $0 < u < R-d$. Under these conditions no intersections are possible so we have

$$p_0(u) = 1,\ p_1(u) = p_2(u) = 0.$$

(ii) $R-d \leqq u < \sqrt{R^2-d^2}$. Now only zero or one intersections are possible. Referring to Figure 2 we see that:

$$p_0(u) = 1 - 2\psi/\pi,\ p_1(u) = 2\psi/\pi,\ p_2(u) = 0,$$

$$\psi = \arccos\left(\frac{R^2 - d^2 - u^2}{2ud}\right).$$

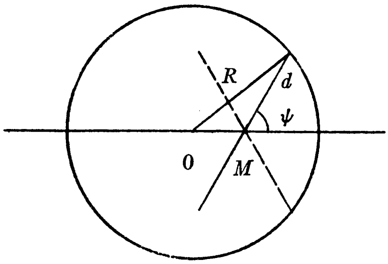


FIG. 2.

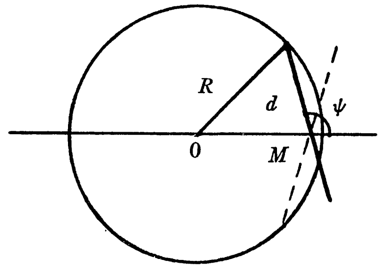


FIG. 3.

(iii) $\sqrt{R^2-d^2} \le u \leqq R$. Only one or two intersections are possible and, referring to Figure 3, we see that:

$$p_0(u) = 0$$
$$p_1(u) = 2 - 2\psi/\pi$$
$$p_2(u) = 2\psi/\pi - 1.$$

As for case (a) we obtain

$$P_A[Z=0] = \frac{\sqrt{R^2-d^2}}{R} - \frac{2}{\pi R}\int_{R-d}^{\sqrt{R^2-d^2}} \arccos\left(\frac{R^2-d^2-u^2}{2ud}\right) du,$$

$$P_A[Z=1] = 2 - \frac{2\sqrt{R^2-d^2}}{R} + \frac{2}{\pi R}\int_{R-d}^{\sqrt{R^2-d^2}} \arccos\left(\frac{R^2-d^2-u^2}{2ud}\right) du$$
$$- \frac{2}{\pi R}\int_{\sqrt{R^2-d^2}}^{R} \arccos\left(\frac{R^2-d^2-u^2}{2ud}\right) du,$$

$$P_A[Z=2] = \frac{\sqrt{R^2-d^2}}{R} - 1 + \frac{2}{\pi R}\int_{\sqrt{R^2-d^2}}^{R} \arccos\left(\frac{R^2-d^2-u^2}{2ud}\right) du.$$

$$P_B[Z=0] = \frac{R^2-d^2}{R^2} - \frac{4}{\pi R^2}\int_{R-d}^{\sqrt{R^2-d^2}} u \arccos\left(\frac{R^2-d^2-u^2}{2ud}\right) du,$$

$$P_B[Z=1] = 2 - \frac{2}{R^2}(R^2-d^2) + \frac{4}{\pi R^2}\int_{R-d}^{\sqrt{R^2-d^2}} u \arccos\left(\frac{R^2-d^2-u^2}{2ud}\right) du$$
$$- \frac{4}{\pi R^2}\int_{\sqrt{R^2-d^2}}^{R} u \arccos\left(\frac{R^2-d^2-u^2}{2ud}\right) du$$

$$P_B[Z=2] = \frac{1}{R^2}(R^2-d^2) - 1 + \frac{4}{\pi R^2}\int_{\sqrt{R^2-d^2}}^{R} u \arccos\left(\frac{R^2-d^2-u^2}{2ud}\right) du.$$

3. Evaluation of the integrals. Let

$$I = \int_a^b \arccos\left(\frac{R^2-d^2-u^2}{2ud}\right) du.$$

To evaluate this integral we integrate by parts and then make, successively, the following substitutions:

$$u = (R + d)\cos\phi$$
$$(R + d)\sin\phi = 2\sqrt{Rd}\sin\theta.$$

After some work we finally get:

$$I = u\arccos\left(\frac{R^2 - d^2 - u^2}{2ud}\right)\Big|_a^b$$
$$- \{(d - R)[F(b_2 \mid k) - F(a_2 \mid k)] - (d + R)[E(b_2 \mid k) - E(a_2 \mid k)]\}$$

where:

$$F(\eta \mid k) = \int_0^\eta (1 - k^2\sin^2\theta)^{-1/2}d\theta$$

$$E(\eta \mid k) = \int_0^\eta \sqrt{1 - k^2\sin^2\theta}\, d\theta.$$

$$k^2 = 4Rd/(R + d)^2$$

$$a_2, b_2 = \theta \text{ limits of integration.}$$

$F(\eta \mid k)$ is the elliptic integral of the 1st kind and $E(\eta \mid k)$ is the elliptic integral of the 2nd kind. Let

$$I' = \int_a^b u\arccos\left(\frac{R^2 - d^2 - u^2}{2ud}\right)du.$$

Then, as for integral I, we obtain:

$$I' = \frac{u^2}{2}\arccos\left(\frac{R^2 - d^2 - u^2}{2ud}\right)\Big|_a^b + R^2(b_2 - a_2) + Rd\sin\theta\cos\theta\Big|_{a_2}^{b_2}.$$

4. Main results. Using the results of Section 3 we can write expressions for $P_A[Z=i]$, $P_B[Z=i]$.

Let:

$$\gamma = d/R$$
$$\theta_1 = \pi/2$$
$$\theta_2 = \arcsin\sqrt{\frac{1+\gamma}{2}}$$
$$\theta_3 = \arcsin\sqrt{\frac{2+\gamma}{4}}.$$

Case A

(i) $0 < \gamma \leqq 1$

$$P_A[Z=0] = \frac{2}{\pi}\{(1-\gamma)[F(\theta_1 \mid k) - F(\theta_2 \mid k)] + (1+\gamma)[E(\theta_1 \mid k) - E(\theta_2 \mid k)]\}$$

$$P_A[Z=1]=\frac{2}{\pi}\arccos\frac{\gamma}{2}$$
$$-\frac{2}{\pi}\{(1-\gamma)[F(\theta_1\mid k)-F(\theta_2\mid k)]+(1+\gamma)[E(\theta_1\mid k)-E(\theta_2\mid k)]\}$$
$$+\frac{2}{\pi}\{(1-\gamma)[F(\theta_2\mid k)-F(\theta_3\mid k)]+(1+\gamma)[E(\theta_2\mid k)-E(\theta_3\mid k)]\}$$
$$P_A[Z=2]=1-(2/\pi)\arccos(\gamma/2)$$
$$-\frac{2}{\pi}\{(1-\gamma)[F(\theta_2\mid k)-F(\theta_3\mid k)]+(1+\gamma)[E(\theta_2\mid k)-E(\theta_3\mid k)]\}.$$

(ii) $$1\leqq\gamma\leqq 2$$

$$P_A[Z=0]=0$$
$$P_A[Z=1]=\frac{2}{\pi}\arccos\left(\frac{\gamma}{2}\right)$$
$$-\frac{2}{\pi}\{(\gamma-1)[F(\theta_1\mid k)-F(\theta_3\mid k)]-(\gamma+1)[E(\theta_1\mid k)-E(\theta_3\mid k)]\}$$
$$P_A[Z=2]=1-\frac{2}{\pi}\arccos\left(\frac{\gamma}{2}\right)$$
$$+\frac{2}{\pi}\{(\gamma-1)[F(\theta_1\mid k)-F(\theta_3\mid k)]-(\gamma+1)[E(\theta_1\mid k)-E(\theta_3\mid k)]\}.$$

Since we can write k in terms of γ, viz., $k^2=4\gamma/(1+\gamma)^2$, each of these expressions can be computed as functions of the single variable γ.

Case B

(i) $$0<\gamma\leqq 1$$

$$P_B[Z=0]=2-\frac{2\gamma}{\pi}\sqrt{1-\gamma^2}-\frac{4}{\pi}\arcsin\sqrt{\frac{1+\gamma}{2}}$$
$$P_B[Z=1]=\frac{4}{\pi}\sqrt{1-\gamma^2}-\frac{\gamma}{4}\sqrt{4-\gamma^2}-2+\frac{2}{\pi}\arccos\left(\frac{\gamma}{2}\right)$$
$$+\frac{8}{\pi}\arcsin\sqrt{\frac{1+\gamma}{2}}-\frac{4}{\pi}\arcsin\sqrt{\frac{2+\gamma}{4}}$$
$$P_B[Z=2]=1+\frac{2\gamma}{\pi}\left(\tfrac{1}{2}\sqrt{4-\gamma^2}-\sqrt{1-\gamma^2}\right)-\frac{2}{\pi}\arccos\left(\frac{\gamma}{2}\right)$$
$$+\frac{4}{\pi}\arcsin\sqrt{\frac{2+\gamma}{4}}-\frac{4}{\pi}\arcsin\sqrt{\frac{1+\gamma}{2}}.$$

(ii) $$1 \leqq \gamma \leqq 2$$

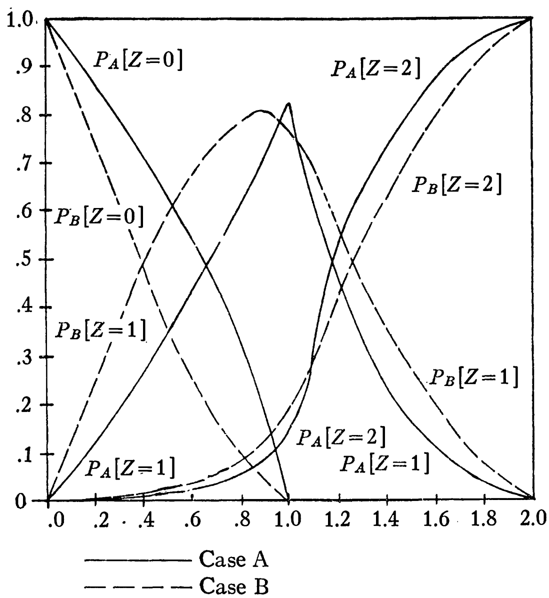
$$P_B[Z=0]=0$$

$$P_B[Z=1]=2-\frac{\gamma}{\pi}\sqrt{4-\gamma^2}+\frac{2}{\pi}\arccos\left(\frac{\gamma}{2}\right)-\frac{4}{\pi}\arcsin\sqrt{\frac{2+\gamma}{4}}$$

$$P_B[Z=2]=\frac{\gamma}{\pi}\sqrt{4-\gamma^2}-1+\frac{4}{\pi}\arcsin\sqrt{\frac{2+\gamma}{4}}-\frac{2}{\pi}\arccos\left(\frac{\gamma}{2}\right).$$

The form of the above results was chosen to make their evaluation by computer easy. In Appendix 1 we give numerical results for values of γ between 0 and 2.

5. Appendix 1.



GRAPH I. This graph shows the relationship between the various probabilities for all γ in [0, 2].

Case A

TABLE 2

γ	$P_A[Z=0]$	$P_A[Z=1]$	$P_A[Z=2]$
.2	.86	.14	.00
.4	.70	.29	.01
.6	.52	.46	.02
.8	.30	.64	.06
1.0	.00	.84	.16
1.2	.00	.45	.55
1.4	.00	.22	.78
1.6	.00	.13	.87
1.8	.00	.04	.96

Case B

TABLE 3

γ	$P_B[Z=0]$	$P_B[Z=1]$	$P_B[Z=2]$
.2	.75	.25	.00
.4	.51	.48	.01
.6	.28	.68	.04
.8	.11	.80	.09
1.0	.00	.78	.22
1.2	.00	.57	.43
1.4	.00	.38	.62
1.6	.00	.21	.79
1.8	.00	.07	.93

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1. C. B. Boyer, A History of Mathematics, Wiley, New York, 1968.
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3. J. J. Sylvester, On Buffon's problem of the needle, Acta Math., 14 (1891) 185–205.

The following references are also relevant:

4. A. L. Clarke, Buffon's needle problem, Canad. J. Res., 9 (1933) 402 and 11 (1934) 658.
5. N. T. Gridgeman, Geometric probability and the number π, Scripta Math., 25 (1960) 183–95.
6. B. C. Kahan, A practical demonstration of a needle experiment to give a number of concurrent estimates for π, J. Roy. Statist. Soc. Ser. A, 124 (1961) 227–39.
7. M. S. Klamkin, On the Uniqueness of the distribution function for the Buffon needle problem, Amer. Math. Monthly, 60 (1953) 677–680.
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9. L. Mantel, An extension of the Buffon needle problem. Ann. Math. Statist., 22 (1951) 314–15, also Ann. Math. Statist., 24 (1953) 674–677.
10. J. F. Ramaley, Buffon's noodle problem, Amer. Math. Monthly, 76 (1969) 916–18.

ON N-SEQUENCES

T. C. BROWN, Simon Fraser University, and MAX L. WEISS,
University of California, Santa Barbara

In [1], it is shown that the Fibonacci sequence 1, 1, 2, 3, 5, $\cdots$ has the property that if any one term is removed from the sequence then every positive integer can be expressed as the sum of some of the terms that remain, and that if any two terms are removed, then there is a positive integer that cannot be expressed as the sum of some of the remaining terms. We describe this situation

Case B

TABLE 3

γ	$P_B[Z=0]$	$P_B[Z=1]$	$P_B[Z=2]$
.2	.75	.25	.00
.4	.51	.48	.01
.6	.28	.68	.04
.8	.11	.80	.09
1.0	.00	.78	.22
1.2	.00	.57	.43
1.4	.00	.38	.62
1.6	.00	.21	.79
1.8	.00	.07	.93

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In [1], it is shown that the Fibonacci sequence 1, 1, 2, 3, 5, $\cdots$ has the property that if any one term is removed from the sequence then every positive integer can be expressed as the sum of some of the terms that remain, and that if any two terms are removed, then there is a positive integer that cannot be expressed as the sum of some of the remaining terms. We describe this situation by saying that removal of any two terms from the Fibonacci sequence renders it *incomplete*, while removal of any one term does not. A sequence which is not incomplete will be called *complete*.

In this note we consider, for each $n=0, 1, 2, \cdots$, nondecreasing sequences which are rendered incomplete by the removal of any $n+1$ terms, but not by the removal of any n terms. We call such a sequence an *n-sequence*. Thus the Fibonacci sequence is a 1-sequence.

We characterize in a simple way the set of all 1-sequences and show that there are no n-sequences for any $n \geqq 2$. A simple description of the set of all 0-sequences seems more difficult, and is left open.

THEOREM 1. *The sequence* $\{a_k\}_{k=1}^{\infty}$ *is a* 1-*sequence if and only if it has the following three properties.:*

$$a_1 = a_2 = 1. \tag{1}$$

$$a_{k+1} = a_k \quad or \quad a_{k+1} = \sum_{j=1}^{k-1} a_j + 1, \qquad k \geqq 2. \tag{2}$$

$$a_{k+1} = \sum_{j=1}^{k-1} a_j + 1 \quad \text{for infinitely many } k. \tag{3}$$

Proof of sufficiency. The proof that the Fibonacci sequence is a 1-sequence given in [1] translates more or less immediately into a proof of the sufficiency of the given conditions, and is given here, slightly altered, for the sake of completeness.

Let $\{b_k\}_{k=1}^{\infty}$ be the sequence obtained from $\{a_k\}_{k=1}^{\infty}$ by deleting one term from $\{a_k\}$. Suppose that N is the smallest number which is not the sum of terms from $\{b_k\}$ and that $a_n \leqq N < a_{n+1}$. Then $\sum_{j=1}^{n-1} b_j \geqq \sum_{j=1}^{n-1} a_j = (\sum_{j=1}^{n-1} a_j+1)-1 = a_{n+1}-1 \geqq N$. Now let L be the smallest number greater than N which is the sum of terms from $\{b_k\}$ and suppose that $L=\sum_{m=1}^{r} b_{j_m}$, where $j_1<j_2< \cdots < j_r \leqq n-1$. Then if $b_{j_1}=1$ we get

$$\sum_{m=2}^{r} b_{j_m} = L - 1 \geqq N,$$

and hence, since N is not the sum of terms from $\{b_k\}$, $L-1>N$, contradicting the choice of L. If $b_{j_1}>1$ then $N \geqq a_n \geqq b_{n-1} \geqq b_{j_1}$, hence $N > b_{j_1}-1 \geqq 1$, hence by the definition of $N, b_{j_1}-1=\sum_{p=1}^{s} b'_{j_p}$, and moreover $b'_{j_p} < b_{j_m}$ for all $m \geqq 2$ and all p. Thus $\sum_{p=1}^{s} b'_{j_p} + \sum_{m=2}^{r} b_{j_m} = L-1 \geqq N$, hence $>N$, again contradicting the choice of L. Thus $\{a_k\}$ remains complete if one term is deleted.

Suppose now that both a_m and a_n are deleted, $m<n$, to obtain $\{b_k\}$. We may assume without loss of generality that $a_{n+1}=\sum_{j=1}^{n-1} a_j+1$. Then

$$\sum_{j=1}^{n-2} b_j = \sum_{j=1}^{n-1} a_j - a_m = a_{n+1} - 1 - a_m < a_{n+1} - 1 = b_{n-1} - 1,$$

hence $b_{n-1}-1$ is not the sum of terms of $\{b_k\}$. Then $\{a_k\}$ is rendered incomplete by the deletion of any two terms.

Proof of necessity. Let $\{a_k\}$ be a 1-sequence. Then clearly $a_1=a_2=1$, and a_3 is either 1 or 2. Suppose that, for $k=1, 2, \cdots, n (n\geqq 2)$, either $a_{k+1}=a_k$ or $a_{k+1}=\sum_{j=1}^{k-1} a_j+1$. We wish to show that then either $a_{n+2}=a_{n+1}$ or $a_{n+2}=\sum_{j=1}^{n} a_j+1$. Clearly $a_{p+2}\leqq \sum_{j=1}^{p} a_j+1$ for all p, otherwise deleting a_{p+1} from $\{a_k\}$ would yield an incomplete sequence. Thus in particular $a_{n+1}\leqq a_{n+2}\leqq \sum_{j=1}^{n} a_j+1$.

Let $\{b_k\}$ be the sequence obtained from $\{a_k\}$ by deleting a_1 and a_{n+1}. We show that if $a_{n+1}<a_{n+2}<\sum_{j=1}^{n} a_j+1$ then $\{b_k\}$ is complete, contradicting the assumption that $\{a_k\}$ is a 1-sequence. Note that $\{b_k\}_{k=1}^{n-1}=\{a_k\}_{k=2}^{n}$ is an initial part of the complete sequence $\{a_k\}_{k=2}^{\infty}$. It is easy to show by induction, and we leave it to the reader to do so, that if $\{c_k\}$ is complete and $N\leqq c_1+\cdots+c_m$ then N is the sum of terms from $\{c_k\}_{k=1}^{m}$. Hence if $N\leqq b_1+\cdots+b_{n-1}$ then N is the sum of terms from $\{b_k\}_{k=1}^{n-1}$ and we say that the sequence $\{b_k\}$ is complete through $b_1+\cdots+b_{n-1}$.

Now

$$b_n = a_{n+2} < \sum_{j=1}^{n} a_j + 1 = 1 + \sum_{j=1}^{n-1} b_j + 1,$$

so $b_n\leqq\sum_{j=1}^{n-1}b_j+1$, and hence $\{b_k\}$ is complete through $b_1+\cdots+b_n$. Next,

$$b_{n+1} = a_{n+3} \leqq \sum_{j=1}^{n+1} a_j + 1 = 1 + b_1 + \cdots + b_{n-1} + a_{n+1} + 1$$
$$< 1 + b_1 + \cdots + b_{n-1} + a_{n+2} + 1 = 1 + b_1 + \cdots + b_n + 1.$$

Thus $b_{n+1}\leqq\sum_{j=1}^{n}b_j+1$, and so $\{b_k\}$ is complete through $b_1+\cdots+b_{n+1}$. Now assume $\{b_k\}$ is complete through $b_1+\cdots+b_{n+p}$, $p\geqq 1$. Then

$$b_{n+p+1} = a_{n+p+3} \leqq \sum_{j=1}^{n+p+1} a_j + 1 = 1 + \sum_{j=2}^{n} a_j + \sum_{j=n+1}^{n+p+1} a_j + 1$$
$$= 1 + b_1 + \cdots + b_{n-1} + \sum_{j=n+1}^{n+p+1} a_j + 1 < 1 + b_1 + \cdots + b_{n-1}$$
$$+ \sum_{j=n+2}^{n+p+2} a_j + 1 = 1 + b_1 + \cdots + b_{n-1} + \sum_{j=n}^{n+p} b_j + 1,$$

hence $b_{n+p+1}\leqq\sum_{j=1}^{n+p}b_j+1$, hence $\{b_k\}$ is complete through $b_1+\cdots+b_{n+p+1}$.

We have now shown that $\{b_k\}$ is complete through $b_1+\cdots+b_m$ for all m, and hence is complete. This completes the proof that condition (2) holds for the sequence $\{a_k\}$. Condition (3) then follows immediately. This completes the proof of Theorem 1.

THEOREM 2. *For $n\geqq 2$, there are no n-sequences.*

Proof. For any $n>2$, an n-sequence can be converted into a 2-sequence by deleting any $n-2$ terms. Hence it is sufficient to show there are no 2-sequences.

Suppose $\{a_k\}$ is a 2-sequence. Then if any term is deleted, the result is a 1-sequence; in particular $\{a_k\}_{k=2}^{\infty}$ is a 1-sequence. But then by Theorem 1 there are arbitrarily large n such that $a_{n+2}=\sum_{j=2}^{n} a_j+1$, or, since $a_1=1$, $a_{n+2}=\sum_{j=1}^{n} a_j$.

But then, choosing n large enough that $a_n > 1$, we have $a_{n+2} - 1 > \sum_{j=1}^{n-1} a_j$, and hence $a_{n+2} - 1$ is not the sum of terms from the sequence resulting when a_n, a_{n+1} are deleted from $\{a_k\}$. Thus $\{a_k\}$ is not a 2-sequence. This completes the proof.

Supported in part by NRC Grant A3983.

Reference

1. Solution by Jack Silver of Problem E1424 proposed by V. E. Hoggatt and Charles King, Amer. Math. Monthly, 68 (1961) 179–180.

COMMENTS ON A TRAJECTORY-INDICATING DEVICE

J. L. BRENNER, University of Arizona

The concept. An acoustic trajectory indicating device is one that infers the path of a supersonic projectile from observations of a very short (10–20 ft.) section of its trajectory. A device of this type might be useful on a hovering helicopter under attack from ground fire. Preliminary tests of a crude model at Stanford Research Institute some time ago did show promise.

The geometry. The operation of such a device can be described in purely geometric terms. Over a short distance, a supersonic projectile travels in a practically straight line. The detectable phenomenon is the shock cone left by the projectile. This is a disturbance in the form of a right circular cone with apex at the bullet, axis along the trajectory. Thus the detectable phenomenon is a right circular conical surface traveling vertex forward along its axis. See Figure 1. A microphone records a crack when a shock hits it. (This is the crack heard by the eardrum when a bullet passes nearby.)

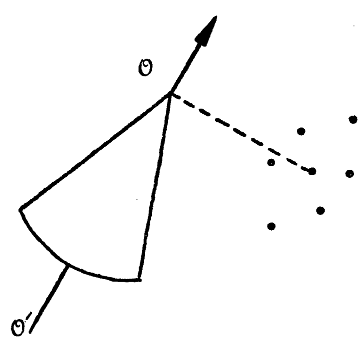

FIG. 1. Trajectory-indicating device. Right circular cone traveling vertex forward along its axis.

The unknowns. To detect the passage and orientation of a trajectory requires a battery of tiny microphones. Each microphone must record the mo-

But then, choosing n large enough that $a_n > 1$, we have $a_{n+2} - 1 > \sum_{j=1}^{n-1} a_j$, and hence $a_{n+2} - 1$ is not the sum of terms from the sequence resulting when a_n, a_{n+1} are deleted from $\{a_k\}$. Thus $\{a_k\}$ is not a 2-sequence. This completes the proof.

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COMMENTS ON A TRAJECTORY-INDICATING DEVICE

J. L. BRENNER, University of Arizona

The concept. An acoustic trajectory indicating device is one that infers the path of a supersonic projectile from observations of a very short (10–20 ft.) section of its trajectory. A device of this type might be useful on a hovering helicopter under attack from ground fire. Preliminary tests of a crude model at Stanford Research Institute some time ago did show promise.

The geometry. The operation of such a device can be described in purely geometric terms. Over a short distance, a supersonic projectile travels in a practically straight line. The detectable phenomenon is the shock cone left by the projectile. This is a disturbance in the form of a right circular cone with apex at the bullet, axis along the trajectory. Thus the detectable phenomenon is a right circular conical surface traveling vertex forward along its axis. See Figure 1. A microphone records a crack when a shock hits it. (This is the crack heard by the eardrum when a bullet passes nearby.)

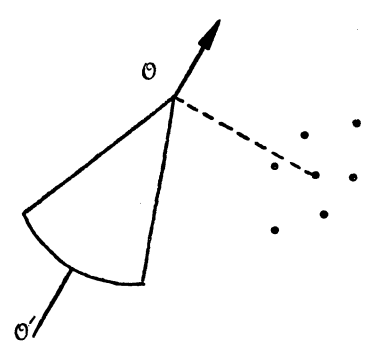

FIG. 1. Trajectory-indicating device. Right circular cone traveling vertex forward along its axis.

The unknowns. To detect the passage and orientation of a trajectory requires a battery of tiny microphones. Each microphone must record the moment the surface of the shock cone sweeps over it. If there are 7 microphones, 6 bits (Δt) of information will be detected. The number of parameters to detect is also 6: 3 coordinates to locate $\mathcal{O}$, and 3 direction angles to describe the orientation of the velocity vector.

The redundancies. Actually not all 3 angles α, β, γ, need to be accurately detected, since $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$. But two of the angles α, β do not wholly determine the third. Further, the trajectory is just as well determined if $\mathcal{O}$ is replaced by any other point $\mathcal{O}'$ on the same trajectory. Practically speaking, these redundancies mean that 7 microphones will usually determine the magnitude as well as the orientation of the velocity vector, in addition to the location of the point $\mathcal{O}$ of closest approach.

The calculations. There remains the substantial task of programming any necessary calculations so that they can be carried out in a short time. The whole idea of a device would be to permit evasive action, or counteraction. A tentative program was developed at SRI, based on certain equations that involve (1) preliminary approximation to the unknown velocity, (2) certain arithmetic operations, (3) iteration toward convergence of the calculations. The articles [1, 2] are based on this work. For some important special cases, they even extend it.

The ambiguous cases. Although the data detected and reported by the microphones should be enough to determine the trajectory for almost all possible orientations, a practical program must fail in some measurable proportion of the cases. Any engineering datum has only finite accuracy, and hence, some ambiguity. With 6 data, all ambiguous, the indicated solution may be so far from the truth as to be useless.

Construction of the model. The Sonics Department of the Institute constructed a prototype detector under the sponsorship of Combat Development and Evaluation Command, U. S. Army. Several tests were made with the detector. Static firing tests showed the feasibility of the device; on two occasions, the detector was mounted on a hovering helicopter, and bullets were fired at various miss distances, 20 to 100 feet. The device had axis 10 feet, so that it could be fitted into a cube 7 feet on a side.

Testing the data. A special computer program was written by Ronald Davis, then a student, to reduce the data. His excellent logic reported failure to obtain some trajectories, particularly at 100-foot miss distance, but in no case did he report the wrong trajectory. In most cases (except for the largest miss distance) a useful approximation of the trajectory was recovered.

Further plans. Further development has been deferred. Possible improvements suggested in the original invention disclosure are (1) a very high-fidelity microphone; (2) a smaller device, (this would require special filtering circuits to detect the N-wave of the shock); (3) development of a miniature airborne computer, to give instant readout of the trajectory.

Advantages of a small device go far beyond compactness. Disturbance from the rotors of a helicopter, or from the heat of its exhaust, vitiates some measurements that are taken on a 10-foot device but not on a miniature device.

Critique. The practical need for a trajectory-indicating device is manifold. In the first place, when an enemy fires from the ground at a hovering helicopter, his position is hard to determine even if he is standing on open terrain and uncamouflaged. The reader will remember that an astronaut flying around the moon was not able to locate his companions on the lunar surface, even though he knew approximately where they were. Usually when an object moves it is easier to detect; however, this may depend on conditions of lighting.

Besides the ambiguities in data mentioned above, there may be false signals, missing signals, or coincidences of the circuitry. False signals can occur from reflections of the shock by helicopter surfaces, and also from refractions caused by temperature gradients around the indicating device.

Complications to the geometry. In the equations originally developed, the trajectory was assumed to be a right circular cone, moving in the direction in which its vertex points, and along its axis. If there is a wind, or if the helicopter is moving, additional variables are introduced, and more microphones will be needed to determine these variables. The shock wave is no longer a right circular cone moving across a stationary array of microphones. The case of a hovering helicopter in a windless atmosphere is, however, one of the practically important cases.

Alternative detection devices. Many engineers have examined, but rejected, the idea of using radar. Although the fundamental reliability of a radar signal is high, only a heavy and expensive machine would be able to look in all directions simultaneously to observe a passing projectile.

Other suggestions regarding acoustic detection have included triangulation of the muzzle blast from the ground. This method simply will not work in practice. The acoustic wave does not progress from the muzzle in a straight line, but its course is disturbed by variations of pressure, temperature, humidity, and most particularly, by even a light wind.

Thus the original passive acoustic device seems to have practical value.

References

1. W. P. Reid, Line of flight from shock recordings, this MAGAZINE, 41 (1968) 59–63.
2. S. J. Zaroodny, Trajectory indicator—a proposal, SIAM Rev., 14 (1966) 1366–1389.

A PURSUIT PROBLEM

GERALD CROUGH, University High School, Los Angeles, California

The curiosity of man seems to be immutable through the passing of generations. Problems which he solved two centuries ago reappear to puzzle and amuse him once again. In this paper I deal with my experience with one such perennial problem; it is:

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A German Shepherd is due north of a Siamese that begins to run east. What curve will the dog trace if he pursues the cat by continuously running toward it?

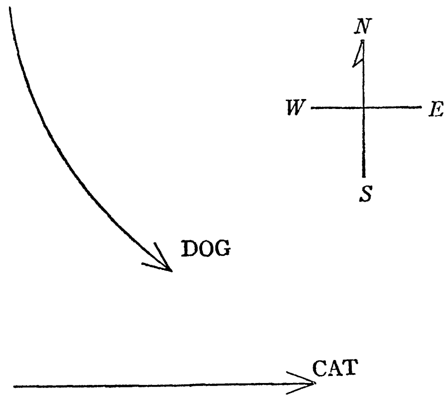

FIG. 1.

I came across this problem in a version that asked only for the distances the animals travel before they meet. I thought my problem was different and never seriously considered previously. My solution follows:

Assign the North-South meridian as the y-axis and the East-West parallel as the x-axis. Assume the cat and dog travel at constant speeds, v and cv, and let the initial distance between them be D_0.

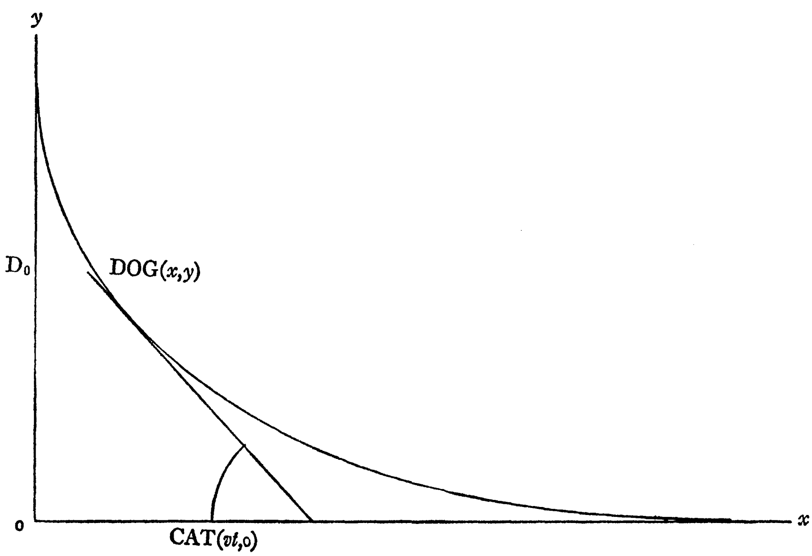

FIG. 2.

Introduce the variable, a, as the angle between the x-axis and the line containing the points the animals occupy. Then write the components of the dog's velocity as

$$\frac{dx}{dt} = cv \cos a \tag{1}$$

$$\frac{dy}{dt} = - cv \sin a \tag{2}$$

Note the presence of the negative sign on the y-component; the y-coordinate decreases as the dog moves.

We begin with the relation

$$\tan a = \frac{y}{vt - x} \cdot \tag{3}$$

Differentiating (3) and using (1), (2) and (3) gives

$$\sec^2 a \frac{da}{dt} = \frac{1}{vt - x} \frac{dy}{dt} - y \frac{v - \frac{dx}{dt}}{(vt - x)^2}$$

$$= \frac{\tan a}{y} (-cv \sin a) - y \cdot \frac{\tan^2 a}{y^2} (v - cv \cos a)$$

and

$$\frac{da}{dt} = - \frac{v}{y} \sin^2 a.$$

It is clear that y is a strictly decreasing function of time and hence has an inverse $t = t(y)$ whose derivative is

$$\frac{dt}{dy} = \frac{1}{dy/dt} = \frac{-1}{cv} \csc a.$$

Then $a = a(t)$ is a function of y and

$$\frac{da}{dy} = \frac{da}{dt} \cdot \frac{dt}{dy} = - \frac{v}{y} \sin^2 a \left(- \frac{1}{cv} \csc a \right)$$

$$= \frac{\sin a}{cy} \cdot$$

Hence

$$\frac{da}{\sin a} = \frac{dy}{cy}$$

and

$$\csc a - \cot a = ky^{1/c}. \tag{4}$$

Substituting

$$\frac{dx}{dy} = \frac{dx}{dt} \cdot \frac{dt}{dy} = -\cot a$$

and $\sqrt{1+(dx/dy)^2} = \csc a$ in (4) and solving for dx/dy gives

$$\frac{dx}{dy} = \frac{k}{2} y^{1/c} - \frac{1}{2k} y^{-1/c}.$$

Hence finally

$$x = \frac{k}{2(1 + 1/c)} y^{1+1/c} - \frac{1}{2k(1 - 1/c)} y^{1-1/c} + A$$

where $A = D_0\, c/c^2 - 1)$ and $k = D_0^{-1/c}$.

However, I found my efforts were not original. In the Advanced Problems section of the August–September 1941 issue of the AMERICAN MATHEMATICAL MONTHLY the problem is given for $c = 2$ and is solved by H. A. Luther.

It is noted there that the problem appears on page 295 of Fine's *Calculus*, and on page 332 of Osgood's *Advanced Calculus*. It also appears as ex. 3 on page 138 of *Elements of Ordinary Differential Equations* by Golomb and Shanks and probably in many other similar textbooks. But what must be the earliest published solution is noted by Arthur Bernhart in Scripta Mathematica, Vol. 20, p. 127: Pierre Bourguer (1698–1758) in a 1732 memoir *Upon New Curves Which May Be Called Lines of Pursuit* [*Lignes de poursuite*, Mémoires de l'Académie Royale des Sciences (1732), pp. 1–14] derives the very same equation I have. I am anticipated by 238 years!

ON GROUP ELEMENTS OF ORDER TWO

M. G. MONZINGO, Southern Methodist University

Exercise 11 on page 11 of [1] asserts that if G is a group and a is the only element of G of order two, then a commutes with every element in G. We prove this assertion and some related results.

LEMMA 1. *If G is a group and a is an element of G of order two, then xax^{-1} has order two, for every x in G.*

Proof. $(xax^{-1})^2 = (xax^{-1})(xax^{-1}) = xa^2x^{-1} = e$. Also, $(xax^{-1})^1 \neq e$ since $a \neq e$.

Proof of the assertion. If a is the only element of order two, then Lemma 1 implies that $xax^{-1} = a$, for every x in G. Hence, $xa = ax$.

Henceforth, the set of elements of G of order two will be denoted by S, and n will denote the number of elements in S.

Substituting

$$\frac{dx}{dy} = \frac{dx}{dt}\cdot\frac{dt}{dy} = -\cot a$$

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Henceforth, the set of elements of G of order two will be denoted by S, and n will denote the number of elements in S.

LEMMA 2. *Let a, b be in S, $a \neq b$; then, ab is in S if and only if $ab = ba$.*

Proof. Suppose that ab is in S. Then, $abab = (ab)^2 = e = a^2b^2 = aabb$. This implies that $ba = ab$. Conversely, $(ab)^2 = abab = a(ab)b = a^2b^2 = e$.

LEMMA 3. *The subgroup generated by S is normal in G.*

Proof. By problem 4 on page 55 of [2] it is sufficient to show that if a is in S and g is in G, then gag^{-1} is in S. This follows from Lemma 1.

THEOREM 1. *If n is nonzero, then n cannot be even.*

Proof. Let $S = \{a_1, a_2, \cdots, a_n\}$, with n even $(n \neq 0)$; then

$$(1) \qquad a_1a_2 = a_ia_1, \qquad a_1a_3 = a_ja_1, \cdots, a_1a_n = a_ka_1,$$

where $\{a_i, a_j, \cdots, a_k\} = \{a_2, a_3, \cdots, a_n\}$. Since $a_1a_t = a_sa_1$ implies $a_1a_s = a_ta_1$, the equations (1) can be paired. Since n is even there is an odd number of these equations and one of the equations is of the form $a_1a_r = a_ra_1$. Without loss of generality, we may suppose that $r = 2$; then $a_1a_2 = a_2a_1$ is in S, and so, by relabeling if necessary, we may (since $a_1, a_2 \neq e$) take $a_3 = a_1a_2$. Then, a_1 commutes with both a_2 and a_3. Now,

$$(2) \qquad a_1a_4 = a_ua_1, \qquad a_1a_5 = a_va_1, \cdots, a_1a_n = a_ka_1.$$

Since again there is an odd number of equations (2), the same reasoning implies that a_1 commutes with at least two more a_q. Since n is finite and a_1 is arbitrary, it follows that all the elements of S commute. By Lemma 2, $S \cup \{e\}$ is closed, and, hence, is a subgroup of G. By Lagrange's theorem, the order of the a_i, 2, divides the order of $S \cup \{e\}$, $n+1$; a contradiction.

LEMMA 4. *If $S = \{a, b, c\}$ and one pair of elements of S commutes then every pair commutes.*

Proof. Assume that $ac = ca$. Then, aba^{-1} must be in S, whence $ab = ba$, and similarly, $bc = cb$.

THEOREM 2. *If $S = \{a, b, c\}$, then G contains either an abelian noncyclic normal subgroup of order 4, or a nonabelian normal subgroup of order 6.*

Proof. If the elements of S commute, then $\{e, a, b, c\}$ is a subgroup. If not then $ac = ba = cb$ and $bc = ab = ca$ so that the subgroup generated by S is $\{e, a, b, c, ab, ba\}$.

The case n odd but neither one nor three is not so nice. For example, in the group of symmetries of the square, S has five elements, four reflections and a half-turn, and two of the reflections do commute with each other but not with all the other elements of S. Of course, if the elements of S commute, then Lemma 2 will apply.

THEOREM 3. *If n is odd $(n \neq 1)$ and the elements of S commute, then $S \cup \{e\}$ is an abelian noncyclic normal subgroup of G.*

If the conditions of Theorem 3 hold for G finite, then $(n+1)$ divides the order of G.

References

1. G. W. Polites, An Introduction to the Theory of Groups, International Textbook, Scranton, 1968.
2. I. N. Herstein, Topics in Algebra, Blaisdell, Waltham, 1964.

AN INTERESTING PROPERTY OF SQUARE MATRICES

S. KESAVAN, Kilpauk, Madras, India

In this short note, we derive with the help of Cramer's Rule an interesting invariant property of square matrices having the same determinant value ($\neq 0$). Starting from this result, we can derive certain well-known properties of square matrices.

NOTATION. If $A=(a_{ij})$, $1 \leqq i, j \leqq n$, is an $n\times n$ matrix, then as usual A_{ij} and $|A|$ will denote the cofactor of a_{ij} and the determinant of A, respectively.

The system of linear equations, $\sum_{j=1}^{n} a_{ij}x_j = b_i$ $(i=1, 2, \cdots, n)$ can be written in the form $AX=B$, where

$$A = (a_{ij}), \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix}.$$

The matrix obtained by replacing the jth column of A by B will yield a square matrix whose determinant will be denoted as D_{B^j}. Now we have the well-known theorem,

CRAMER'S RULE. *If $AX=B$ is a system of n linear equations in n unknowns, with $|A| \neq 0$, then,*

$$x_j = \frac{D_{B^j}}{|A|}.$$

We are now in a position to pass on to our main theorem.

THEOREM. *Let B be a fixed $(n\times 1)$ matrix. Then,*

$$A \begin{bmatrix} D_{B^1} \\ D_{B^2} \\ \cdot \\ \cdot \\ \cdot \\ D_{B^n} \end{bmatrix}$$

yields the same $(n\times 1)$ matrix for all $(n\times n)$ matrices A having the same determinant value ($\neq 0$).

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yields the same $(n \times 1)$ matrix for all $(n \times n)$ matrices A having the same determinant value ($\neq 0$).

Proof. Choose an $(n\times 1)$ matrix X, such that $AX=B$. Then

I $$X = A^{-1}B.$$

Applying Cramer's rule, we get

II $$X = \frac{1}{|A|}\begin{bmatrix} D_{B^1} \\ D_{B^2} \\ \cdot \\ \cdot \\ \cdot \\ D_{B^n} \end{bmatrix}.$$

I and II imply

III $$A\begin{bmatrix} D_{B^1} \\ D_{B^2} \\ \cdot \\ \cdot \\ \cdot \\ D_{B^n} \end{bmatrix} = |A|\cdot B.$$

Since $|A|$ and B are fixed, the left hand side of III is also fixed. This completes the proof of the theorem.

A few consequences of this result can now be seen. We will consider a particular case of this general result. Taking $A=(a_{ij})$, $A'=\text{diag}(|A|, 1, 1, \cdots, 1)$ and

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix},$$

we get

i. $|A|=|A'|\neq 0$.
ii. $D_{Bj}=A_{ij}$ in A.
iii. $D'_{B^1}=1$ and $D'_{Bj}=0$ for $2\leqq j\leqq n$ in A'.

Applying the above theorem,

$$A\begin{bmatrix} A_{11} \\ A_{12} \\ \cdot \\ \cdot \\ \cdot \\ A_{1n} \end{bmatrix} = A'\begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \text{or,} \quad A\begin{bmatrix} A_{11} \\ A_{12} \\ \cdot \\ \cdot \\ \cdot \\ A_{1n} \end{bmatrix} = \begin{bmatrix} |A| \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix},$$

which gives us

$$\sum_{j=1}^{n} a_{ij}A_{1j} = |A| \quad \text{if } i = 1, \quad \text{and}$$
$$= 0 \qquad \text{if } i \neq 1.$$

This principle can be extended and by interchange of rows we can apply this case to get the well-known and important property of nonsingular square matrices,

$$\sum_{k=1}^{n} a_{ik}A_{jk} = \delta_{ij}\,|\,A\,| \quad \text{where } \delta_{ij} = 1 \quad \text{or 0 according as } i = j \text{ or } i \neq j.$$

Thus the theorem not only establishes a connection between nonsingular matrices whose determinants are equal but also leads to the above-mentioned properties of square matrices in an elegant manner.

ANSWERS

A513. The squares of any three numbers in arithmetic progression with common difference d satisfy the equation $x-2y+z=2d^2$. For example, $a^2-2(a+d)^2+(a+2d)^2=2d^2$. Consequently, the four vertices are coplanar, so the volume of the tetrahedron is zero.

It follows that the a_i need not be all in one sequence, but merely that the common difference of the triads be the same, d.

A514. It may be concluded that S has exactly three elements. If P is as given, then not-P would be, "At most one element of S is in T." Now if S has s elements we may restate not-P as, "At least $s-1$ elements of S are not in T." But if this is equivalent to the not-P given then $s-1=2$ and it follows that $s=3$.

A515. The number of diagonals is $2^{n-1}>n$, the number of dimensions, if $n>3$. Hence the diagonals cannot be mutually perpendicular. The inequality is easily established by induction on n beginning at $n=3$, for example. The diagonals bisect each other for all $n\geqq 2$ as is easily shown by either vector methods or analytical geometry methods.

A516. From the inequality between the arithmetic and geometric means, we have

$$\frac{c+\sqrt{ab}+\sqrt{ab}}{3} \geqq \sqrt[3]{c\sqrt{ab}\sqrt{ab}} = \sqrt[3]{abc}$$

Therefore

$$c \geqq 3\sqrt[3]{abc} - 2\sqrt{ab}$$

A517. The given double sum is equivalent to

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^n$$

which is the value of the nth difference column constructed from the function given by $f(x)=x^n$ evaluated at $0, 1, 2, \cdots n$. This value is well known to be $n!$

(Quickies on page 114.)

ANTI-ISOMORPHISMS vs. ISOMORPHISMS

LYLE E. PURSELL, University of Missouri-Rolla

Let A and B be semigroups and let ϕ be a one-to-one mapping from A onto B. If $\phi(ab)=\phi(a)\phi(b)$ for all a, b in A, then ϕ is called an *isomorphism*, but if $\phi(ab)=\phi(b)\phi(a)$ for all a, b in A, then ϕ is called an *anti-isomorphism*. If $A=B$, then ϕ is called respectively an *automorphism* or an *antiautomorphism*. Most textbooks on abstract algebra treat isomorphisms extensively, but either ignore anti-isomorphisms or mention them only perfunctorily. This is unfortunate since many natural mappings between concrete semigroups are anti-isomorphisms.

We list below several interesting and perhaps useful properties of anti-isomorphisms on semigroups. These results are quite easy to prove and could be assigned as exercises in a beginning abstract algebra course. In most cases the proofs are omitted since they are a straightforward application of the definitions involved. These results can be extended to rings in an obvious way.

Let $I_*[A, B]$ be the set of all isomorphisms from A onto B, let $I^*[A, B]$ be the set of all anti-isomorphisms from A onto B, and let

$$I[A, B] = I_*[A, B] \cup I^*[A, B].$$

The following are immediate:

(1) $\phi \in I_*[A, B]$ and $\psi \in I_*[B, C] \Rightarrow \psi \circ \phi \in I_*[A, C]$.

(2) $\phi \in I_*[A, B]$ and $\psi \in I^*[B, C] \Rightarrow \psi \circ \phi \in I^*[A, C]$.

(3) $\phi \in I^*[A, B]$ and $\psi \in I_*[B, C] \Rightarrow \psi \circ \phi \in I^*[A, C]$.

(4) $\phi \in I^*[A, B]$ and $\psi \in I^*[B, C] \Rightarrow \psi \circ \phi \in I_*[A, C]$.

(5) $\phi \in I_*[A, B] \Rightarrow \phi^{-1} \in I_*[B, A]$.

(6) $\phi \in I^*[A, B] \Rightarrow \phi^{-1} \in I^*[B, A]$.

(7) THEOREM. *If A and B are groups and are anti-isomorphic (i.e., $I^*[A, B] \neq \emptyset$), then they are isomorphic (i.e., $I_*[A, B] \neq \emptyset$).*

Proof. Let $\alpha \in I^*[A, B]$ and define $\beta: A \to A$ by $\beta(a)=a^{-1}$ for all a in A. Then $\beta \in I^*[A, A]$. By (4) $\alpha \circ \beta \in I_*[A, B]$.

If A and B are commutative semigroups, then $I[A, B]=I_*[A, B]=I^*[A, B]$. Henceforth we will assume A and B are semigroups which are not commutative.

(8) THEOREM. *If A is not commutative, i.e., there exist a, b in A such that $ab \neq ba$, then*

$$I_*[A, B] \cap I^*[A, B] = \emptyset.$$

Proof. $\phi \in I_*[A, B] \Rightarrow c=\phi(a)\cdot\phi(b)=\phi(a\cdot b) \Rightarrow \phi^{-1}(c)=ab$. $\phi \in I^*[A,B] \Rightarrow c=\phi(a)\cdot\phi(b)=\phi(b\cdot a) \Rightarrow \phi^{-1}(c)=ba$. But $ab \neq ba$.

In the remainder of this paper we will assume $B=A$. One can easily show:

(9) THEOREM. *$I[A, A]$ with the operation of composition is a group having $I_*[A, A]$ as a normal subgroup.*

(10) THEOREM. *The index of* $I_*[A, A]$ *in* $I[A, A]$ *is either* 1 (*if* $I^*[A, A]=\varnothing$) *or* 2 (*if* $I^*[A, A]\neq\varnothing$).

Proof. If $I^*[A, A]=\varnothing$, then the index is 1.

Suppose $I^*[A, A]\neq\varnothing$. Let γ, δ be any two antiautomorphisms in $I^*[A, A]$. By (6), δ^{-1} is in $I^*[A, A]$. Hence, by (4), $\gamma \circ \delta^{-1}$ is in $I_*[A, A]$.

(Note. A special version of this result is given by C. C. MacDuffee, *An Introduction to Abstract Algebra*, John Wiley & Sons, Inc., 1940 or Dover Publications, Inc., 1966, p. 257. For a quaternion algebra over a real field he points out that conjugation is an antiautomorphism, hence, "the determination of all automorphisms carries with it the determination of all antiautomorphisms, and conversely.")

If A is a group, then $I^*[A, A]\neq\varnothing$, since the mapping $(a\rightarrow a^{-1}): A\rightarrow A$ is an antiautomorphism. We do not know whether or not there exist semigroups A such that $I^*[A, A]$ is empty.

Another question which we have not been able to answer is: Does there exist a noncommutative semigroup A and a one-to-one mapping α from A onto A such that $\alpha(ab)\in\{\alpha(a)\alpha(b), \alpha(b)\alpha(a)\}$ for all a, b in A, but $\alpha(ab)=\alpha(a)\alpha(b)\neq\alpha(b)\alpha(a)$ and $\alpha(cd)=\alpha(d)\alpha(c)\neq\alpha(c)\alpha(d)$ for some a, b, c, d in A?

LOCAL AND UNIFORM LIPSCHITZIANISM

W. G. DOTSON, JR., North Carolina State University

In a recent textbook on intermediate mathematical analysis [1, p. 141, Exercise 1] the student is asked to "Prove the following: If f satisfies a Lipschitz condition at each point of a closed interval $[a, b]$, then f satisfies a uniform Lipschitz condition on $[a, b]$." The author gives a very plausible hint to proceed as one usually does in proving that continuity at each point of a closed interval $[a, b]$ implies uniform continuity on $[a, b]$. It should be pointed out, however, that all attempts to prove the above proposition are doomed to certain failure. The following counterexample, while quite elementary, may be of interest because of its indirectness and freedom from messy details.

Consider the well-known function $f(x)=x\,\sin(\pi/x)$, $0<x\leqq 1$, $f(0)=0$. For each x in $(0, 1]$ $f'(x)$ exists, and hence f satisfies a Lipschitz condition at each such x, e.g., see [1, p. 119, Theorem 6.3]. For $0\leqq x\leqq 1$ we clearly have $|f(x)-f(0)|\leqq 1\cdot|x-0|$, so that f also satisfies a Lipschitz condition at 0. But if f satisfied a uniform Lipschitz condition on $[0, 1]$, then it would follow immediately that f is of bounded variation on $[0, 1]$, and, of course, it is well known that this is not true.

This function also makes a nice example of a uniformly continuous function (on a closed interval) which satisfies no uniform Lipschitz condition.

Reference

1. A. E. Labarre, Jr., Intermediate Mathematical Analysis, Holt, Rinehart and Winston, New York, 1968.

(10) THEOREM. *The index of* $I_*[A, A]$ *in* $I[A, A]$ *is either* 1 (*if* $I^*[A, A]=\emptyset$) *or* 2 (*if* $I^*[A, A]\neq\emptyset$).

Proof. If $I^*[A, A]=\emptyset$, then the index is 1.

Suppose $I^*[A, A]\neq\emptyset$. Let γ, δ be any two antiautomorphisms in $I^*[A, A]$. By (6), δ^{-1} is in $I^*[A, A]$. Hence, by (4), $\gamma \circ \delta^{-1}$ is in $I_*[A, A]$.

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EDITED BY D. ELIZABETH KENNEDY, University of Victoria

Materials intended for review should be sent to: Professor D. Elizabeth Kennedy, Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada.

Reviews of texts at the freshman-sophomore level based upon classroom experience will be welcomed by the Book Review Editor.

A boldface capital C in the margin indicates a classroom review.

C *Fundamentals of Trigonometry*. By Earl W. Swokowski. Prindle, Weber, & Schmidt, Boston, 1968. 211 pp.

This book is a well-organized and clearly written introduction to trigonometry, intended for the student who is about to begin calculus. The material covered is about right for a 2 credit one-semester course, or a 3 credit one-quarter course.

The book begins with a review of real numbers and the Cartesian coordinate system, together with an introduction to functions and graphing. The first chapter is quite detailed, but may be omitted if students are well prepared in general college algebra.

The trigonometric functions are introduced through the device of the "wrapping function" $P(t)$. $P(t)$ is the point on the unit circle whose distance from (1, 0) measured counterclockwise along the circumference is t. In fact $P(t) = (\cos t, \sin t)$, where t is given in radians. This function is shown to be periodic with period 2π. This development is rather difficult for beginning students, but once mastered lays a strong foundation for study of the various properties of sine and cosine, particularly periodicity and the effect of phase-shift.

Angles, degree measure, and triangles are introduced only after the basic properties of the trigonometric functions have been developed. The use of tables and interpolation are presented at this point.

The presentation of identities, conditional equations, and multiple-angle formulae appear to be standard, and the exercises are of the appropriate level of difficulty. There is a section on the inverse trigonometric functions, and substantial material on topics such as angular velocity and harmonic motion.

The solution of triangles via Law of Sines and Law of Cosines is preceded by a discussion of logarithms, which is useful but may be unnecessary for the well-prepared student.

The final chapter introduces complex numbers through operations on ordered pairs (a, b) where (0, 1) is identified with i. The treatment of complex numbers is remarkably complete for a text at this level in that it even includes an elementary proof of DeMoivre's theorem. The discussion of roots of unity may be perhaps too advanced for the average student.

The appendix contains the usual tables (logs, trig functions, and logs of trig functions) plus the answers to all odd-numbered problems. A supplementary pamphlet containing the answers for the even exercises is also available.

G. S. GLAZER, University of Wisconsin, Waukesha

MU ALPHA THETA MATHEMATICS BOOKLIST

The National High School and Junior College Mathematics Club, Mu Alpha Theta, has revised and updated its list of *Enrichment Mathematics Books for School and Public Libraries.* Single copies of the 1970 list may be obtained without charge by sending a self-addressed stamped number 10 (business size) envelope to: High School Book List, Mu Alpha Theta, Mathematics Department, University of Oklahoma, 1000 Asp Avenue, #215, Norman, Oklahoma 73069.

Persons desiring quantities of the list are encouraged to send $10 per hundred copies to help with the cost of the project.

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. SUTHERLAND FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems proposed. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before September 15, 1971.

PROPOSALS

788. *Proposed by R. S. Luthar, University of Wisconsin, Waukesha.*

Show that $(p_1 \cdot p_1 \cdot p_3 \cdot \,\cdots\, p_n+1)^{2^k}-1$ has at least $n+k$ distinct prime divisors, where $p_1, p_2 \cdots p_n$ are the first n primes.

789. *Proposed by Edwin P. McCravy, Midlands Technical Education Center, Columbia, South Carolina.*

Show that this "loot begging" cryptarithm

$$\begin{array}{r} S\ E\ N\ D \\ M\ O\ R\ E \\ \hline M\ O\ N\ E\ Y \end{array}$$

has exactly $\binom{b-8}{2}$ solutions in the base b.

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790. *Proposed by Stanley Rabinowitz, Far Rockaway, New York.*

(1) Find all triangles ABC such that the median to side a, the bisector of angle B, and the altitude to side c are concurrent.

(2) Find all such triangles with integral sides.

791. *Proposed by D. Rameswar Rao, Secunderabad, India.*

Show that the only solution in positive integers of $x^3+y^3+z^3=u^3$ with x, y, z, u in arithmetic progression is $x=3$, $y=4$, $z=5$, and $u=6$.

792. *Proposed by Murray S. Klamkin, Ford Scientific Laboratory.*

It is a known result that a necessary and sufficient condition for a triangle inscribed in an ellipse to have a maximum area is that its centroid coincide with the center of the ellipse. Show that the analogous result for a tetrahedron inscribed in an ellipsoid is not valid.

793. *Proposed by Gregory Wulczyn, Bucknell University.*

Find the determinant of lowest order with entries from the interval $-1\leqq x \leqq +1$whose value is 2^{38}.

794. *Proposed by Leon Bankoff, Los Angeles, California.*

If I, O, H are the incenter, circumcenter and orthocenter, respectively, of a triangle ABC in which $C>B>A$, show that I must lie within the triangle BOH.

795. *Proposed by L. Carlitz, Duke University.*

Define $\{B_n\}$ by means of $B_0=1$ and $\sum_{k=0}^{n} \binom{n}{k} B_k=B_k$ for $n>1$. Show that for arbitrary m, $n\geqq 0$,

$$(-1)^m \sum_{r=0}^{m} \binom{m}{r} B_{n+r} = (-1)^n \sum_{s=0}^{n} \binom{n}{s} B_{m+s}.$$

SOLUTIONS

Late Solutions

Ronald Dubson, University of Illinois: **751**; *Alfred Kohler, Long Island University:* **750, 753**; *J. Ernest Wilkins, Jr., Howard University:* **755**; *Rina Rubenfeld, New York City Community College, Brooklyn:* **754.**

A Cryptarithmic Trick

761. [May, 1970] *Proposed by John Hudson Tiner, High Ridge, Missouri.*

Solve the cryptarithm

$$\begin{array}{r} C\ H\ U\ C\ K \\ T\ R\ I\ G\ G \\ T\ U\ R\ N\ S \\ \hline T\ R\ I\ C\ K\ S \end{array}$$

Solution by J. A. H. Hunter, Toronto, Canada.

$$\begin{array}{cccccc} & C & H & U & C & K \\ & T & R & I & G & G \\ & T & U & R & N & S \\ \hline T & R & I & C & K & S \end{array}$$

$T=1$. If $C=6$, then $R=$zero, and 2 "carry" is needed from HRU. But maximum value of $(H+R+U$ $+$"carry") would be 19, so "Carry 2" from HRU is impossible.

Hence, possibles comprise:

$$\begin{array}{ll|c|ll} C=9 & & 8 & & 7 \\ R=2 & 3 & 0 & 2 & 0 \end{array}$$

Now, $K+G=10$.

Continue by tabulating above C, R values, with corresponding values of K, G, then N, and I and U, finally H. This does not entail any undue amount of numerical working.

"Impossibles" being struck out when they occur, we are left with $C=9$, $R=2$, $K=4$, $G=6$, $N=8$, $U=5$, $I=$zero, $H=3$. Since $T=1$, there remains $S=7$.

The complete layout gives $93594+12066+15287=120947$.

Also solved by Andrew N. Aheart, West Virginia State College; Miguel Bamberger, Monterey, California; Jeffrey H. Baumwell, Brooklyn, New York, Joseph V. Bello, Philadelphia, Pennsylvania; A. J. Berlau, Hartsdale, New York; Haig Bohigian, John Jay College of Criminal Justice, New York; Stanley Chow, University of British Columbia; Gerald C. Dodds, JRB-Singer, Inc., State College, Pennsylvania; Marlies Gerber, Zephyrhills High School, Florida; Louise S. Grinstein, New York, New York; Philip Haverstick, Fort Belvoir, Virginia; Claudia Hulce, Battle Creek, Michigan; Alfred Kohler, Long Island University; Henrietta O. Midonick, New York, New York; William Miesslein, Madison, Wisconsin; Joseph V. Michalowicz, Catholic University of America; John W. Milson, Butler County Community College, Pennsylvania; Otto Mond, Suffern, New York; Steve Morfey, University of British Columbia; George A. Novacky, Jr., University of Pittsburgh; Walter J. Parlicek, 64 Finance Section, APO, San Francisco; E. F. Schmeichel, College of Wooster, Ohio; F. Max Stein, Colorado State University; Paul Sugarman, Massachusetts Institute of Technology; Charles W. Trigg, San Diego, California; William G. Varnell, Jacksonville University, Florida; John R. Ventura, Jr., U. S. Naval Underwater Weapons Research and Engineering Station, Rhode Island; R. Wardrop, Central Michigan University; Kenneth M. Wilke, Topeka, Kansas; A. M. Widrowicz, Sioux City, Iowa; K. L. Yocom, University of Wyoming; and the proposer. Two incorrect solutions were received.

Varnell and *Anthony Balmarcich*, St. Joseph's College, Pennsylvania, noted that if all of the components are allowed to come from the set consisting of the first 10 digits then the following is also a solution:

$$\begin{array}{cccccc} & 4 & 7 & 3 & 4 & 8 \\ & 0 & 5 & 6 & 2 & 2 \\ & 0 & 3 & 5 & 1 & 9 \\ \hline 0 & 5 & 6 & 4 & 8 & 9 \end{array}$$

Prime Pairs

762. [May, 1970] *Proposed by Arthur Marshall, Madison, Wisconsin.*

Let n be a natural number greater than three. Prove that there exist two odd primes p_1 and p_2 such that

$$2n \equiv p_1 + p_2 \pmod{p_2}.$$

I. *Solution by Shiv Kumar, Panjabi University, and Miss Nirmal, Government Girls' High School, Panipat, India.*

Something more than the required can be proved. In fact if n is any natural number there exist two odd primes p_1 and p_2 such that $n \equiv p_1 \pmod{p_2}$. Take some natural number n. Choose some odd prime p_1 such that $n - p_1$ is not of the form $2^\alpha p_1^\beta$. When this is done, factor $n - p_1$. It will contain some odd prime factor p_2. Then $n \equiv p_1 \pmod{p_2}$. For example $1 \equiv 11 \pmod 5$, $2 \equiv 7 \pmod 5$, $3 \equiv 13 \pmod 5$, $4 \equiv 11 \pmod 7$, $5 \equiv 11 \pmod 3$, etc.

II. *Solution by E. P. Starke, Plainfield, New Jersey.*

Dirichlet's well-known theorem on primes in an arithmetic progression implies much more. Let p_2 be *any* odd prime which does not divide n; the theorem guarantees that there are *infinitely many* terms in the arithmetic progression $2n + kp_2$, $k = 1, 2, \cdots$, which are prime. Let p_1 be any one of them. Then $p_1 = 2n + kp_2$ which immediately implies $2n \equiv p_1 \equiv p_1 + p_2 \pmod{p_2}$.

Also solved by Walter Blumberg, New Hyde Park, New York; Richard L. Breisch, Pennsylvania State University; John L. Brown, Jr., Pennsylvania State University; Thomas K. Cooper, Hershey, Michigan; Dean Hickerson, Robert F. Jackson, University of Toledo, Ohio; Alfred Kohler, Long Island University; Lew Kowarski, Morgan State College, Maryland; George A. Novacky, Jr., University of Pittsburgh; S. Ron Olivier, Sioux City, Iowa; Walter J. Pavlicek, 64 Finance Section, APO San Francisco; Bob Prielipp, Wisconsin State University at Oshkosh; John E. Prussing, University of Illinois; Marilyn Rodeen, San Mateo, California; E. F. Schmeichel, College of Wooster, Ohio; G. P. Speck, Bradley University, Illinois; Philip Tracy, Liverpool, New York; Stephen Turner, University of Illinois; Kenneth M. Wilke, Topeka, Kansas; Stephen E. Wilson, Flagstaff, Arizona; K. L. Yocom, University of Wyoming; and the proposer.

Quasi Zeta Functions

763. [May, 1970] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Prove:

$$\left(1+\frac{1}{3^{10}}+\frac{1}{5^{10}}+\cdots\right)=\left(1+\frac{1}{3^4}+\frac{1}{5^4}\cdots\right)\left(1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\cdots\right).$$

Solution by M. G. Greening, University of New South Wales, Australia.

$\zeta(s) = \sum_{s=1}^{\infty} n^{-s}$, convergent for $|s| > 1$.

$$1 + 3^{-4} + 5^{-4} + \cdots = \zeta(4) - \frac{1}{2^4}\zeta(4) = 15\zeta(4)/2^4$$

$$1 - 2^{-6} + 3^{-6} - 4^{-6} + \cdots = \zeta(6) - 2\left(\frac{1}{2^6}\zeta(6)\right) = (1 - 2^{-5})\zeta(6)$$

$$1 + 3^{-10} + 5^{-10} + \cdots = \zeta(10) - \frac{1}{2^{10}}\zeta(10) = (1 - 2^{-5})\frac{33}{22}\zeta(10).$$

As $\zeta(2n) = 2^{2n-1} B_n \pi^{2n}/(2n)!$ where B_n is the nth Bernoulli number, and $B_2 = 1/30$, $B_3 = 1/42$, $B_5 = 5/66$, the result follows after simplification.

Also solved by Bernard August, Glassboro State College, New Jersey; Miguel Bamberger, Monterey, California; Walter Blumberg, New Hyde Park, New York; Wray G. Brady, Slippery Rock State College, Pennsylvania; Richard L. Breisch, Pennsylvania State University; Donald R. Childs, Naval Underwater Weapons Research and Engineering Station, Rhode Island; Gerald C. Dodds, HRB-Singer, Inc., State College, Pennsylvania; D. Dummit, San Mateo High School, California; Louise Grinstein, New York, New York; Jeffrey Hoffstein, Bronx High School of Science, New York; Robert F. Jackson, University of Toledo, Ohio; Shiv Kumar, Panjabi University, and Miss Nirmal, Government Girls' High School, Panipat, India (jointly); J. R. Kuttler, Johns Hopkins Applied Physics Laboratory, Maryland; Herbert R. Leifer, Pittsburgh, Pennsylvania; Michael J. Martino, IBM, Poughkeepsie, New York; Kenneth Rosen, University of Michigan; L. E. Schaefer, General Motors Institute, Flint, Michigan; E. F. Schmeichel, College of Wooster, Ohio; E. P. Starke, Plainfield New Jersey; Paul D. Thomas, Naval Research Laboratory, Washington, D.C.; Graham C. Thompson, Binghamton, New York; Michael R. Wise, University of Colorado; Gregory Wulczyn, Bucknell University; K. L. Yocom, University of Wyoming; and the proposer.

A Nonsingular Matrix

764. [May, 1970] *Proposed by F. D. Parker, St. Lawrence University.*

Let $A=[a_{ij}]$ be a nonsingular square matrix, and denote its determinant by $d(A)$. If the same nonzero number x is added to each element of A to produce the matrix $A+x=[a_{ij}+x]$ then $d(A+x)=d(A)$ if and only if the sum of the elements of A^{-1} is zero.

Solution by Carl D. Meyer, Jr., North Carolina State University.

This is an immediate consequence of the stronger statement

$$d(A+xJ)=d(A)(1+xj^TA^{-1}j)$$

where $j^T=[1\ 1\ \cdots\ 1]$ and $J=jj^T$. This identity follows directly from the fact that the determinant is an additive function of its columns. Using this fact repeatedly yields

$$\begin{aligned} d(A)+xJ) = d(A) &+ d([xj, a_2, \cdots, a_n]) \\ &+ d([a_1, xj, a_3, \cdots, a_n]) + \cdots \\ &+ d([a_1, \cdots, a_{n-1}, xj]) \end{aligned}$$

where the a_j's denote columns of A. (All other determinants obtained are zero because they contain two or more columns equal to xj). Furthermore,

$$d([a_1, \cdots, a_{i-1}, xj, a_{i+1}, \cdots, a_n]) = x\sum_p A_{pi}$$

so that

$$d(A+xJ)=d(A)+x\sum_{p,q}A_{pq}=d(A)+xj^T(adjA)j=d(A)(1+xj^TA^{-1}j)$$

where the A_{pq}'s denote cofactors of A and $adjA$ denotes the transpose of the matrix of cofactors of A. (Note that if $d(A)=0$ then $d(A+xJ)=xj^T(adjA)j$).

Also solved by Marjorie R. Bicknell, Sunnyvale, California; Walter Blumberg, New Hyde Park, New York; Derrill J. Bordelon, Naval Underwater Weapons Research and Engineering Station, Rhode Island; Robert DeCarli, Rosary Hill College, Buffalo, New York; Ronald Dubson, University of Illinois; M. G. Greening, University of New South Wales, Australia; Philip Haverstick, Fort Belvoir, Virginia; Robert F. Jackson, University of Toledo, Ohio; Alfred Kohler, Long Island Uni-

versity; Shiv Kumar, Panjabi University, and Miss Nirmal, Government Girls' High School. Panipat, India (jointly); J. R. Kuttler, Johns Hopkins Applied Physics Laboratory, Maryland; Joseph V. Michalowicz, Catholic University of America; C. Bruce Myers, Austin Peay State University; Warren Page, New York City Community College; Albert J. Patsche, Rock Island Arsenal, Illinois; Simeon Reich, Israel Institute of Technology, Haifa, Israel; E. F. Schmeichel, College of Wooster, Ohio; G. P. Speck, Bradley University, Illinois; E. P. Starke, Plainfield, New Jersey; Philip Tracy, Liverpool, New York; Gregory Wulczyn, Bucknell University; K. L. Yocom, University of Wyoming; and the proposer.

For other properties of $d(A+x)$ Bicknell referred to her article, *The lambda number of a matrix: the sum of its N^2 cofactors*, Amer. Math. Monthly, 72, 3, March, (1965) 260–4.

Trisecting Cluster Points

765. [May, 1970] *Proposed by Stanley Rabinowitz, Far Rockaway, New York.*

Let ABC be an isosceles triangle with right angle at C. Let $P_0=A$, $P_1=$the midpoint of BC, $P_{2k}=$the midpoint of AP_{2k-1}, and $P_{2k+1}=$the midpoint of BP_{2k} for $k=1, 2, 3, \cdots$ Show that the cluster points of the sequence $\{P_n\}$ trisect the hypotenuse.

I. *Solution by John Oman, Wisconsin State University.*

The problem may be generalized by letting ABC be any triangle and showing the cluster points trisect AB. The sequences $\{P_{2k}\}$ and $\{P_{2k+1}\}$ are infinite subsets of the compact set ABC and hence have cluster points. Let P' and P'' be any cluster points of P_{2k} and P_{2k+1}, respectively. Since $P_{2k}P_{2k-1}=AP_{2k}=\frac{1}{2}AP_{2k-1}$ and $P_{2k}P_{2k+1}=BP_{2k+1}=\frac{1}{2}BP_{2k}$, it follows by continuity that $P'P''=AP'=\frac{1}{2}AP''$ and $P'P''=BP''=\frac{1}{2}BP'$. Hence A, P', and P'' are collinear, with P' the midpoint of AP'', and B, P', P'' are collinear, with P'' the midpoint of BP'. Hence P' and P'' are the trisection points of AB and the unique cluster points of $\{P_{2k}\}$ and $\{P_{2k+1}\}$, respectively. Since every cluster point of $\{P_k\}$ is also a cluster point of at least one of the sets $\{P_{2k}\}$ or $\{P_{2k+1}\}$ the result now follows.

II. *Solution by Emmett D. Kinkade, Garfield High School, Seattle, Washington.*

The result is true for any triangle ABC.

Let $C=0$, $\overrightarrow{CB}=\overrightarrow{OB}=B$ and $\overrightarrow{CA}=\overrightarrow{OA}=A$.

If each P_i is expressed as a linear combination of the vectors B and A, i.e.,

$$P_i = xA + yB$$

then

$$P_1 = \tfrac{1}{2}B = 0\cdot A + \tfrac{1}{2}B$$

$$P_2 = \tfrac{1}{2}P_1 + \tfrac{1}{2}A = \tfrac{1}{2}A + \tfrac{1}{4}B$$

$$P_3 = \tfrac{1}{2}P_2 + \tfrac{1}{2}B = \tfrac{1}{4}A + \tfrac{5}{8}B, \text{ etc.}$$

In general for P_{2k} the x's form the sequence $1/2, 5/8, 21/32, \cdots, x_n$ where

$$x_n = \frac{\sum_{t=0}^{n-1} 4^t}{2\cdot 4^{n-1}} = \frac{4^n-1}{6\cdot 4^{n-1}} = \frac{1}{6}\left(4-\frac{1}{4^{n-1}}\right)$$

and $\lim_{n\to\infty} x_n = 2/3$.

The y's form the sequence

$$\frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \cdots, y_n = \frac{4^{n-1}}{3 \cdot 4^n}$$

with $\lim_{n\to\infty} y_n = 1/3$.

Thus the P_{2k} approach $\frac{2}{3}A + \frac{1}{3}B$ one of the points of trisection of $\overline{BA}$.

Similarly for P_{2k+1},

$$x_n = \frac{4^{n-1} - 1}{3 \cdot 4^{n-1}} \to \frac{1}{3}$$

and

$$y_n = \frac{1}{6}\left(4 - \frac{1}{4^{n-1}}\right) \to \frac{2}{3}$$

so that the P_{2k+1} approach $\frac{1}{3}A + \frac{2}{3}B$, the other point of trisection of $\overline{BA}$.

Also solved by Walter Blumberg, New Hyde Park, New York; Derrill J. Bordelon, Naval Underwater Weapons Research and Engineering Station; Wray G. Brady, Slippery Rock State College, Pennsylvania; Richard L. Breisch, Pennsylvania State University; Stanley Chow, University of British Columbia; William F. Fox, Moberly Junior College, Missouri; Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Australia; Jeffrey Hoffstein, Bronx High School of Science, New York; Robert F. Jackson, University of Toledo, Ohio; Emmett D. Kinkade, Garfield High School, Seattle, Washington (second solution); Lew Kowarski, Morgan State College, Maryland; Shiv Kumar, Panjabi University, and Miss Nirmal, Government Girls' High School, Panipat, India (jointly); J. R. Kuttler, Johns Hopkins Applied Physics Laboratory, Maryland; Henrietta O. Midonick, New York; Steve Morfey, University of British Columbia; George A. Novacky, Jr., University of Pittsburgh; Marilyn Rodeen, Balboa High School, San Francisco, California; E. F. Schmeichel, College of Wooster, Ohio; G. P. Speck, Bradley University, Illinois; E. P. Starke, Plainfield, New Jersey; Philip Tracy, Liverpool, New York; Zalman Usiskin, University of Chicago; Gregory Wulczyn, Bucknell University, Pennsylvania; Robert L. Young, Cape Cod Community College, Massachusetts; and the proposer.

Similar Nilpotent Matrices

766. [May, 1970] *Proposed by Warren Page, New York City Community College.*

Let N_1 and N_2 be two $n \times n$ nilpotent matrices over the field F. If N_1 and N_2 have the same nullity k and the same minimal polynomial p, what is the largest value of n for which N_1 and N_2 must be similar?

Solution by E. F. Schmeichel, College of Wooster, Ohio.

Let M_k denote the $k \times k$ matrix having 1's just above the main diagonal and 0's elsewhere, and consider the 7×7 block diagonal matrices

$$N_1 = \begin{bmatrix} M_3 & & \\ & M_3 & \\ & & M_1 \end{bmatrix} \quad \text{and} \quad N_2 = \begin{bmatrix} M_3 & & \\ & M_2 & \\ & & M_2 \end{bmatrix}.$$

Note that N_1 and N_2 both have nullity 3 and minimal polynomial x_3, and that they are not similar since their invariants are different (Herstein, *Topics in Algebra*, p. 253). Since $7=3+3+1=3+2+2$ is the smallest integer with two distinct partitions having the same number of parts and the same largest part, it follows that 6 is the largest value of n for which N_1 and N_2 must be similar.

Also solved by Derrill J. Bordelon, Naval Underwater Systems Center, Newport, Rhode Island; Joseph V. Michalowicz, Catholic University of America; and the proposer.

A Property of Medians

767. [May, 1970] *Proposed by Harry Sitomer, C. W. Post College, New York.*

Given triangle ABC. Let A', B' and C' be in open segments BC, CA and AB, respectively, such that AA', BB' and CC' are concurrent at G and $AG/GA' = BG/GB' = CG/GC'$.

Prove that AA', BB' and CC' are medians of ABC. How does the conclusion change if "open segments" is replaced by "lines"?

Solution by E. P. Starke, Plainfield, New Jersey.

We take all segments to be directed segments: thus $AG=-GA$, etc. Now B, A' and C are collinear points, one on each side (extended if necessary) of triangle AGB'. By Menelaus' theorem we have

$$\frac{AA'}{A'G}\cdot\frac{GB}{BB'}\cdot\frac{B'C}{CA} = -1. \tag{2}$$

Similarly, from triangle CGB' and collinear points C', B and A, we have

$$\frac{CC'}{C'G}\cdot\frac{GB}{BB'}\cdot\frac{B'A}{AC} = -1. \tag{3}$$

By adding unity to each member of $AG/GA' = CG/GC'$ from (1) we have

$$\frac{AA'}{GA'} = \frac{CC'}{GC'}. \tag{4}$$

An easy combination of (2), (3), (4) gives

$$CB' = B'A. \tag{5}$$

Since these are directed segments (5) means that B' is the midpoint of AC. Similarly, by selecting other triangles, we obtain C' as midpoint of AB, and A' as midpoint of BC.

So (1) implies that A', B', C' lie in the open segments BC, CA, AB, respectively, and the "open segments" restriction in the hypothesis is unnecessary.

In order for any other conclusion to hold, we must interpret the symbols in (1) to mean undirected segments. Then, upon replacing them by directed segments we see that either all ratios have the same sign—the case already discussed—or two have one sign and the third the opposite sign. Suppose, say,

$$\frac{AG}{GA'} = \frac{CG}{GC'} = -\frac{BG}{GB'}. \tag{6}$$

By the original argument, the first equation implies (5) so that B' is the midpoint of AC. From $AG/GA' = -BG/GB'$, we have

$$\frac{AG}{GA'} + \frac{GA'}{GA'} - 1 = -\frac{BG}{GB'} - \frac{GB'}{GB'} + 1,$$

whence

$$\frac{AA'}{GA'} = 2 - \frac{BB'}{GB'}. \tag{7}$$

Now (5) implies $B'C/CA = -1/2$, whence (2) becomes

$$\frac{AA'}{A'G} \cdot \frac{GB}{BB'} = 2. \tag{8}$$

Putting $GB/BB' = GB'/BB' - 1$ in (8) and eliminating $AA'/A'G$ from (7) and (8) we obtain

$$2r^2 - r + 1 = 0, \tag{9}$$

where $r = GB'/BB'$. But the roots of (9) are imaginary, so that this case cannot hold. Thus the case of medians is the only conclusion.

This is Problem 2, p. 39, Adler, *Modern Geometry*.

Also solved by Derrill J. Bordelon, Naval Underwater Systems Center, Newport, Rhode Island; Wray G. Brady, Slippery Rock State College, Pennsylvania; Mannis Charosh, Brooklyn, New York; Michael Goldberg, Washington, D.C.; M. G. Greening, University of New South Wales, Australia; Shiv Kumar, Panjabi University, and Miss Nirmal, Government Girls' High School, Panipat, India (jointly); Steve Morfey, University of British Columbia; Lawrence A. Ringenberg, Eastern Illinois University; Marilyn Rodeen, Balboa High School, San Francisco, California (partially); E. F. Schmeichel, College of Wooster, Ohio; John R. Ventura, Jr., Naval Underwater Weapons Research and Engineering Station, Newport, Rhode Island; and the proposer.

Comment on Q353

Q353. [January, 1965] Solve the functional equation $f(x+y)f(x-y) = \{f(x)+f(y)\}\{f(x)-f(y)\}$ given f has a second derivative.

[*Submitted by Murray S. Klamkin*]

Comment by Sid Spital, California State College, Hayward.

Answer A353 is correct but fails to point out that all solutions must be odd. This results from first setting $x=0$ in the functional equation: $f(y)f(-y) = f(0)^2 - f(y)^2$; and then setting $y=0$: $f(0)=0$. Hence $f(-y) = -f(y)$.

Comment on Q486

Q486. [September, 1970] Prove that for all n, $(n^5 - n)$ is divisible by 30.

[*Submitted by Robert S. Hatcher*]

Comment by Charles Ziegenfus, Madison College, Harrisonberg, Virginia.

Since it is obvious that $n^5 \equiv n \pmod 2$ and $n^5 \equiv n \pmod 3$ and that the Euler-Fermat theorem for congruences implies that $n^5 \equiv n \pmod 5$ and that 2, 3, and

5 are all relatively prime, it follows that $n^5 \equiv n (\text{mod } 2\cdot 3\cdot 5)$.

Additionally, the following proof was given by a student (Sheryl B. Tadlock) many years ago in a sophomore number theory course:

To show that $n^5 - n$ is divisible by 5, we note that:

$$\begin{aligned} n^5 - n &= n(n+1)(n-1)(n^2+1) \\ &= n(n+1)(n-1)(n^2-5n+6+5n-5) \\ &= n(n+1)(n-1)[(n-2)(n-3)+5(n-1)] \\ &= n(n+1)(n-1)(n-2)(n-3)+5(n)(n+1)(n-1)^2. \end{aligned}$$

Clearly 5 divides both members of the last line and hence $n^5 - n$ is divisible by 5.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q513. Twelve numbers, a_i, are in arithmetic progression such that $a_k + d = a_{k+1}$. Find the volume of the tetrahedron with vertices (a_1^2, a_2^2, a_3^2), (a_4^2, a_5^2, a_6^2), (a_7^2, a_8^2, a_9^2), $(a_{10}^2, a_{11}^2, a_{12}^2)$.

[*Submitted by Charles W. Trigg*]

Q514. Let S and T be sets. If P is the statement, "At least two elements of S are in T" and not-P is the statement, "At least two elements of S are not in T," what conclusions may be drawn?

[*Submitted by Stephen E. Wilson*]

Q515. It is well known that the diagonals of a square bisect each other and are perpendicular to each other. Is the same true for a cube? What is the situation in n-space if $n > 3$?

[*Submitted by Frank J. Papp*]

Q516. Show that if a, b, and c are any positive real numbers, then

$$c \geqq 3\sqrt[3]{abc} - 2\sqrt{ab}.$$

[*Submitted by Norman Schaumberger*]

Q517. Prove that

$$\sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i+j} \binom{n}{i} \binom{n}{j} n^{n-i} j^i = n!.$$

[*Submitted by Richard A. Gibbs*]

(Answers on page 101.)

This principle can be extended and by interchange of rows we can apply this case to get the well-known and important property of nonsingular square matrices,

$$\sum_{k=1}^{n} a_{ik}A_{jk} = \delta_{ij}\,|\,A\,| \quad \text{where } \delta_{ij} = 1 \quad \text{or 0 according as } i = j \text{ or } i \neq j.$$

Thus the theorem not only establishes a connection between nonsingular matrices whose determinants are equal but also leads to the above-mentioned properties of square matrices in an elegant manner.

ANSWERS

A513. The squares of any three numbers in arithmetic progression with common difference d satisfy the equation $x-2y+z=2d^2$. For example, $a^2-2(a+d)^2+(a+2d)^2=2d^2$. Consequently, the four vertices are coplanar, so the volume of the tetrahedron is zero.

It follows that the a_i need not be all in one sequence, but merely that the common difference of the triads be the same, d.

A514. It may be concluded that S has exactly three elements. If P is as given, then not-P would be, "At most one element of S is in T." Now if S has s elements we may restate not-P as, "At least $s-1$ elements of S are not in T." But if this is equivalent to the not-P given then $s-1=2$ and it follows that $s=3$.

A515. The number of diagonals is $2^{n-1}>n$, the number of dimensions, if $n>3$. Hence the diagonals cannot be mutually perpendicular. The inequality is easily established by induction on n beginning at $n=3$, for example. The diagonals bisect each other for all $n\geqq 2$ as is easily shown by either vector methods or analytical geometry methods.

A516. From the inequality between the arithmetic and geometric means, we have

$$\frac{c+\sqrt{ab}+\sqrt{ab}}{3} \geqq \sqrt[3]{c\sqrt{ab}\sqrt{ab}} = \sqrt[3]{abc}$$

Therefore

$$c \geqq 3\sqrt[3]{abc} - 2\sqrt{ab}$$

A517. The given double sum is equivalent to

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^n$$

which is the value of the nth difference column constructed from the function given by $f(x)=x^n$ evaluated at $0, 1, 2, \cdots n$. This value is well known to be $n!$

(Quickies on page 114.)